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Research article

Arbitrary (k, l) States-Solutions of the Dirac and Schrödinger Equations Interacting with Improved Spatially-Dependent Mass Coulomb Potential with an Improved Coulomb-Like Tensor Interaction Model for H-atoms from 3D-RNCS and 3D-NRNCS Symmetries

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The deformed Dirac equation has been investigated, in the context of 3D-relativistic noncommutative space (3D-RNCS) symmetries, using the improved spatially dependent mass Coulomb potential with an improved Coulomb-like tensor interaction (ISDM(CP-CLTI)) model under the conditions of spin symmetry and pseudospin symmetry. The ISDM(CP-CLTI) model is the combining the spatially dependent mass Coulomb potential with the Coulomb-like tensor interaction (CP-CLTI) and the two central terms that are generated to the topological defects of spacespace. Within the confines of the parametric Bopp shift method and conventional perturbation theory, the new relativistic and non-relativistic energy eigenvalues for the hydrogen atoms (H-atoms), such as He^{+}, Li^{+2}, and Be^{3+} under the ISDM(CP-CLTI) model have been derived. The novel values $E_{nc}^{sp}(n, C, m_0, m_1, H, \Theta, \tau, \chi, j, l, s, m)$ and $E_{nc}^{ps}\left(n, C, m_0, m_1, H, \Theta, \tau, \chi, j, l, \tilde{l}, \tilde{s}, \tilde{m}\right)$ that we discovered examined to be dependent on the noncommutativity parameters (NP) (Θ, σ, χ), mixed potential depths $C/q, m_0, m_1, H)$, and quantum atomic discrete quantum numbers $(j, k, l, s, m, \tilde{l}, \tilde{s}, \tilde{m})$. We have obtained several interesting special examples within the framework of relativistic extended quantum mechanics, which we believe will be of interest to the expert researcher. We were able to retrieve the typical results of relativistic and non-relativistic examples in the literature when we applied the three simultaneous constraints $(\Theta, \sigma, \chi) \to (0, 0, 0)$. Compared to previous models that are known from the literature, our new model had novel physical characteristics.

1. INTRODUCTION

Among the most important physical facts known to researchers in various fields of physics and chemistry is that systems interacting with strong fields are described by applying the Klein-Gordon (KG) and Duffin-Kemmer-Petiau (DKP) wave equations for particles with spin zero and integer values, respectively. For particles with half spin such as the quark, antiquark, electrons, and positrons, the researchers agreed to apply the well-known Dirac equation in the literature. Furthermore, the Dirac equation (DE) can be generalized to describe nuclear, atomic, and plasma systems. In arbitrary dimensions, the bound states of the KG and Dirac equations with Coulomb-like scalar plus vector potentials have been investigated in many works. Gu *et al.* obtained exact solutions for the DE with a Coulomb potential and presented the energy levels and the corresponding fine structure in the generalized (D+1) space-time.[1] In 2003, Dong studied the (D+1)-dimensional DE with the Coulomb potential following the Tricomi equation approach and he showed that the energy levels *E(n, l, D)* are dependent on the continuous dimension D[2]. In the next year, Ma *et al.* studied the D-dimensional KG equation with a Coulomb plus scalar potential in higher-dimensional field theory and obtained analytically the eigenfunctions, which are expressed by the confluent hypergeometric functions.[2] Dong *et al.* studied the D-dimensional KG equation with a Coulomb potential and analytically obtained and expressed the eigenfunctions by the confluent hypergeometric function.[3] Hamzavi *et al.* (2010) used an asymptotic iteration method with an arbitrary spin-orbit coupling number k to solve the DE for spatially-dependent mass (SDM) Coulomb potentials $m(r) = m_0 + \frac{m_1}{r}$ (see Equation (3)), including a Coulomb-like tensor potential $U(r) = -\frac{H}{r}$, in the pseudospin symmetry limit, and obtained the energy eigenvalues and corresponding eigenfunctions.[4] For any arbitrary spin-orbit j state. Ikhdair and Ramazan investigated the ef-

fect of a spatially dependent mass function on the solution of the DE with the Coulomb potential in the (3+1), the analytic bound state energy eigenvalues and the corresponding upper and lower two-component spinors of the two Dirac particles were found in closed form using the Nikiforov-Uvarov (NU) approach in the context of spin and pseudo-spin symmetry.[5] Ikhdair calculated the exact bound-state energy eigenvalues by studying the effect of spatially dependent mass functions on the solution of the Klein-Gordon (KG) equation in the (3+1)-dimensions for spinless bosonic particles with mixed scalar-vector Coulomb-like field potentials and masses that are directly proportional and inversely proportional to the distance from the force center. The NU approach is also used to obtain the related KG wave functions for mixed scalar vector and pure scalar Coulomb-like field potentials.[6] Adorno *et al.* studied the relativistic energy levels of a hydrogen-like atom under DE with Coulomb field in the framework of θ-modified, due to space non-commutativity, they showed the degeneracy of levels $2S_{1/2}$, $2P_{1/2}$, and $2P_{3/2}$ is lifted fully in the non-commutative space (NC), allowing new transition channels to emerge.[7] Kupriyanov calculated the hydrogen atom spectrum on curved noncommutative space defined by commutation relations (see Eq. (5)) and demonstrated that the noncommutativity contribution appears as a correction to the fine structure.[8] Chair and Dalabeeh used both the second-order correction of perturbation theory and the exact computation due to Dalgarno-Lewis to compute the second-order noncommutative Stark effect in the ground-state energy of the H-atoms in noncommutative space in an external electric field, and derived the sum rule of the mean oscillator strength.[9] Very recently, Nagiyev *et al.* (2022), in the context of position-dependent mass, extended an exactly solvable model of a nonrelativistic quantum linear harmonic oscillator to the case where an external homogeneous gravitational field is applied.[10] Within the noncommutative quantum electrodynamics theory, Chaichian *et al.* calculated the energy levels of the hydrogen atom as well as the Lamb shift,[11,12] in addition to other works which treated the hydrogen atom-influenced Coulomb potential with constant mass in noncommutative quantum mechanics (NCQM) symmetries.[13,14] The study of quantum theories in modified spaces described by noncommutative coordinates has recently reawakened attention.[15-17] The physical consequences of noncommutativity NC of space coordinates ($\widehat{x}_{\mu}^{(s,h,i)} * \widehat{x}_{\nu}^{(s,h,i)} \neq \widehat{x}_{\nu}^{(s,h,i)} * \widehat{x}_{\mu}^{(s,h,i)}$ and $\widehat{p}_{\mu}^{(s,h,i)} * \widehat{p}_{\nu}^{(s,h,i)} \neq x\widehat{p}_{\nu}^{(s,h,i)} * \widehat{p}_{\mu}^{(s,h,i)}$) have attractive to the attention of specialized researchers to justify the rigorous study of noncommutative versions of quantum field theory and ordinary quantum mechanics. The geometrical characteristics of the space-time under consideration are influenced by the presence of phase-space deformation. As a result, the topological properties influence a quantum system's physical characteristics, including the relativistic and non-relativistic energy eigenvalues that are the subject of this research. Because the eigenvalue solutions are changed by the topological defects in comparison to the results obtained in flat space, it is vital and significant to investigate quantum mechanical systems in the non-relativistic and relativistic limits under these defects. Currently, researchers put great hope in NCQM symmetry in its expanded form to find solutions to many difficult problems that the relativistic and non-relativistic quantum mechanics failed to solve, and we mention some exclusive examples such as the divergence problem in the standard model, the possibility of quantizing gravity, and the problem of unifying it with the rest of the fundamental interactions, etc.[14-22] The concept of quantum mechanics with extensions to deformation noncommutative phase-space is not a new one; Snyder introduced it decades ago in 1947,[23] and Connes introduced its geometric analysis in (1991 and 1994).[24,25] Seiberg and Witten[26] obtained a new version of gauge fields in noncommutative gauge theory by extending earlier ideas on the advent of noncommutative geometry in string theory with a nonzero B-field. One of the potential goals of NC deformation of space-space and phase-phase[27] is to eliminate the observed unwanted divergences or infinities that appear to cause short-range in field theories such as gravitational theory by generating new quantum fluctuations in the domain of nano-scales. I think that this study will advance our understanding of elementary particles and subatomic-scale research. The lack of documentation of the improved SDM Coulomb potential with a Coulomb-like tensor interaction (ISDM(CP-CLTI)) model with a deformed Dirac equation (DDE) for H-atoms (He^{+}, Li^{+2}, Be^{3+}) in the literature provided the impetus for the present study. The vector $V_{sc}\left(\hat{r}\right)$, and scalar $S_{sc}\left(\hat{r}\right)$ potentials of the ISDM(CP-CLTI) models which we are in the process of studying and scrutinizing are presented as follows

$$\begin{aligned}&\left(V_{sc}\left(\hat{r}\right),S_{sc}\left(\hat{r}\right)\right)\\&=\left(V_{sc}(r),S_{sc}(r)\right)\\&\quad-\left(\frac{1}{2r}\frac{\partial\left(V_{sc}(r),S_{sc}(r)\right)}{\partial r}\begin{cases}\mathbf{L\Theta} & \text{for spin symmetry}\\ \tilde{\mathbf{L}}\mathbf{\Theta} & \text{for p-spin symmetry}\end{cases}\right)\\&\quad+O\left(\Theta^{2}\right)\end{aligned}\tag{1}$$

In addition to the SDM which noted with $m_{sc}\left(\hat{r}\right)$, in 3D-RNCS symmetry, expressed as

$$\begin{aligned}&m_{sc}\left(\hat{r}\right)=m(r)\\&\quad-\left(\frac{1}{2r}\frac{\partial m_{sc}(r)}{\partial r}\begin{cases}\mathbf{L\Theta} & \text{for spin symmetry}\\ \tilde{\mathbf{L}}\mathbf{\Theta} & \text{for p-spin symmetry}\end{cases}\right)\\&\quad+O\left(\Theta^{2}\right)\end{aligned}\tag{2}$$

Where $(V_{sc}(r),S_{sc}(r),m(r))$ are the vector and scalar potentials according to the view of 3D-relativistic quantum mechanics (RQM) known in the literature[4,5] are of the form

$$\begin{cases}V_{sc}(r)=-\dfrac{C}{r}=-\dfrac{\hbar c q_{v}}{r}\\ S_{sc}(r)=-\dfrac{C_{s}}{r}=-\dfrac{\hbar c q_{s}}{r}\\ m(r)=m_{0}+\dfrac{m_{1}}{r}\end{cases}\tag{3}$$

where $C=Ze^{2}$, C_{s} is a scalar constant, m_{0} is the integration constant that represents the static mass of the fermion particle, $m_{1}$is the perturbed mass, $m_{1}=m_{0}\lambda_{0}b$, $\lambda_{0}=\frac{\hbar}{m_{0}c}$, λ_{0} is the Compton-like wavelength in FM units, q_{v} is a vector dimensionless real parameter coupling constant. The constant mass $b$is a dimensionless real constant that should be set to zero, while ($\hat{r}$ and r) representing the distances between the two particles in the 3D-RNCS and 3D-RQM symmetries, respectively. The physical behavior of the mass field $m(r)$ is distinguished by two boundary states

that correspond to both short and long distances according to the equation

$$m(r) \approx \begin{cases} \frac{m_1}{r} & \text{At short-range} \\ m_0 & \text{At long-range} \end{cases}$$

The two couplings ($\mathbf{L\Theta}$,$\widetilde{\mathbf{L}}\mathbf{\Theta}$) are just the ordinary scalar products of the typical angular momentum operators $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ with components (L_x, L_y, L_z) and $\left(\widetilde{L}_x, \widetilde{L}_y, \widetilde{L}_z\right)$, and the NC-vector $\mathbf{\Theta} \equiv \overrightarrow{\Theta}$, whose constituents represent the NC-elements' parameter $(\theta_{12}/2, \theta_{23}/2, \theta_{13}/2)$. The noncentral generators in the case of the noncommutative quantum group can be correctly represented as self-adjoint differential operators ($\widehat{x}_{\mu}^{(s,h,i)}$,$\widehat{p}_{\nu}^{(s,h,i)}$) existing in 3 varieties. The first one is the canonical structure (CS), the second one is the Lie structure (LS), and the last one, which satisfies a deformed algebra of type, corresponds to the quantum plane (QP) in the representations of Schrödinger, Heisenberg, and interaction pictures.[28-40]

$$\begin{aligned} \left[x_{\mu}^{(s,h,i)}, p_{\nu}^{(s,h,i)}\right] = i\delta_{\mu\nu} \Rightarrow \left[\widehat{x}_{\mu}^{(s,h,i)}, \widehat{p}_{\nu}^{(s,h,i)}\right]_{*} \\ = i\hbar_{eff}\delta_{\mu\nu} \end{aligned} \tag{4}$$

and

$$\begin{aligned} \left[x_{\mu}^{(s,h,i)}, x_{\nu}^{(s,h,i)}\right] = 0 \Rightarrow \left[\widehat{x}_{\mu}^{(s,h,i)}, \widehat{x}_{\nu}^{(s,h,i)}\right]_{*} \\ = \begin{cases} i\theta_{\mu\nu} : \theta_{\mu\nu} \in IC \text{ For CS}, \\ if_{\mu\nu}^{\alpha}\widehat{x}_{\alpha}^{(s,h,i)} : f_{\mu\nu}^{\alpha} \in IC \text{ For LS}, \\ iC_{\mu\nu}^{\alpha\beta}\widehat{x}_{\alpha}^{(s,h,i)}\widehat{x}_{\beta}^{(s,h,i)} : C_{\mu\nu}^{\alpha\beta} \in IC \text{ For QP}. \end{cases} \end{aligned} \tag{5}$$

(we have applied the natural units $\hbar = c = 1$), the generalized coordinates $\widehat{x}_{\mu}^{(s,h,i)}$ and the corresponding generalizing coordinates $\widehat{p}_{\mu}^{(s,h,i)}$ are $\left(\widehat{x}_{\mu}^{s}, \widehat{x}_{\mu}^{h}, \widehat{x}_{nc\mu}^{i}\right)$ and $\left(\widehat{p}_{\mu}^{s}, \widehat{p}_{\mu}^{h}, \widehat{p}_{\mu}^{i}\right)$ in the 3D-RNCS symmetries, while $x_{\mu}^{(s,h,i)} = \left(x_{\mu}^{s}, x_{\mu}^{h}, x_{\mu}^{i}\right)$ and $p_{\mu}^{(s,h,i)}$=(p_{μ}^{s},p_{μ}^{h},p_{μ}^{i}) are corresponding values in the 3D-RQM symmetries and ICdenoting the complex number field. The usual uncertainty relation corresponding to the LHS of Eq. (5) will be extended to become four uncertainties in the new form symmetries as follows

$$\left|\Delta\widehat{x}_{\mu}^{(s,h,i)}\Delta\widehat{p}_{\nu}^{(s,h,i)}\right| \geq \frac{\hbar_{eff}\delta_{\mu\nu}}{2} \tag{6}$$

and[37-41]

$$\left|\Delta\widehat{x}_{\mu}^{(s,h,i)}\Delta\widehat{x}_{\nu}^{(s,h,i)}\right| \geq \begin{cases} |\theta_{\mu\nu}|/2 : \text{For CS-Symmetry}, \\ h_{\mu\nu}/2 : \text{For LS-Symmetry}, \\ G_{\mu\nu}/2 : \text{For QP-Symmetry}. \end{cases} \tag{7}$$

with $h_{\mu\nu}$ and $G_{\mu\nu}$are, respectively, equal to the average values $\left|\left\langle f_{\mu\nu}^{\alpha}\widehat{x}_{\alpha}^{(s,h,i)}\right\rangle\right|$ and $\left|\left\langle C_{\mu\nu}^{\alpha\beta}\widehat{x}_{\alpha}^{(s,h,i)}\widehat{x}_{\beta}^{(s,h,i)}\right\rangle\right|$. It should be noted that the only uncertainty relation that is acknowledged to exist within the context of three-dimensional relativistic quantum mechanics symmetries that are known in the literature is given by

$$\left|\Delta x_{\mu}^{(s,h,i)}\Delta p_{\nu}^{(s,h,i)}\right| \geq \frac{\hbar\delta_{\mu\nu}}{2}$$

It should be noted that generated uncertainty relation in Eq. (6) from the generalization of the left-hand side of Eq. (4) to the right-hand side form. The second new three-uncertainties relations in Eq. (7) are produced by the effect of deformation of space-space arising from the right-hand side of Eq. (5) that is divided into three varieties. The novel uncertainty relations in Eq. (7) within the context of 3D-RQM symmetries have no equivalents in the literature. To take into account both Heisenberg and interaction pictures, the modified equal-time noncommutative canonical commutation relations (METNCCCRs) were expanded in 3D-RNCS symmetry. Here $\theta_{\mu\nu}$are elements of antisymmetric (3×3) real matrices, the effective Planck constant $\hbar_{eff}$ approximatively equal to reduced Planck constant $\hbar$, $\theta_{\mu\nu} = \varepsilon_{\mu\nu}\theta$ (θ is the infinitesimal parameter NP, and $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} = 1$ for $\mu \neq \nu$ and $\varepsilon_{\varepsilon\varepsilon} = 0$) is just an antisymmetric number while $\delta_{\mu\nu}$is the Kronecker symbol. The Weyl-Moyal star product of $h(x)$ and $g(x)$ $functions$ is $h(x) * g(x)$, which is generalized to the new deformed form by as follows,[38-48]

$$h(x) * g(x) = \exp\left(i\varepsilon^{\mu\nu}\theta\partial_{\mu}^{x}\partial_{\nu}^{x}\right)h(x)g(x) \tag{8}$$

which can be reduced to in the first order of NP θ as follows[46-55]

$$h(x) * g(x) \approx\ h(x)g(x) - \frac{i\varepsilon^{\mu\nu}\theta}{2}\partial_{\mu}^{x}h(x)\partial_{\nu}^{x}g(x)\Big|_{x^{\mu}=x^{\nu}} \tag{9}$$

The sum indices (μ, ν) equal 1,2 and 3. Physically, the second term in Eq. (9) is the influence of space-space deformation. Additionally, to the previously mentioned research related to our research, we mention for example the contribution of Akcay[56] related to the study of the Dirac equation with scalar and vector quadratic potentials and a Coulomb-like tensor potential and other works.[57-60] It is important to point out that we have treated this potential in the framework of extended quantum mechanics symmetries of the modified Klien-Gordon equation[61] (MKGE) framework for bosonic particles and antiparticles. The following is a summary of the current paper's structure. In this context, it is useful for the researcher and the reader to mention that the SDM problem has received wide attention for several decades. We mention, for example, the references.[62-64] As far as the spin or pseudospin symmetry, Wei and his coauthors have worked out some results, e.g.[65,66] The first Section of the paper presents the purpose and scope of our research, and the remaining sections are organized as follows: Section 2 gives an overview of the DE with an SDM Coulomb potential and a Coulomb-like tensor interaction. Section 3 is devoted to studying the deformed Dirac equation (DDE) using the well-known Bopp shift method to obtain the ISDM (CP-CLTI) model's effective two potentials of the ISDM (CP-CLTI) model. Furthermore, using standard perturbation theory, we find the expectation values of the radial terms ($\frac{1}{r^3}$ and $\frac{1}{r^4}$) to calculate the corrected relativistic energy generated by the effect of the perturbed effective potentials ($\Sigma_{pert}^{sc}(r)$and $\Delta_{pert}^{sc}(r)$) of the ISDM(CP-CLTI) model, and we derive the global corrected energies $E_{nc}^{ps}(n, C, m_0 m_1, H, \Theta, \tau, \chi, j, l, s, m)$ and $E_{nc}^{ps}(n, C, m_0, m_1,\ H, \Theta, \tau,\ \chi, j, \tilde{l}, \tilde{s}, \widetilde{m})$ for H-atoms (He^{+}, Li^{+2}, Be^{3+}) under the ISDM (CP-CLTI) model. In the next Sect., we will address some special cases of physical importance to researchers and readers. In Section 5, we study a nonrelativistic limit of spin symmetry and apply our findings to hydrogenic atoms. A brief conclusion is given in Sect. 5.

2. REVIEW OF DE UNDER SDM(CP-CLT) IN 3D-RELATIVIC QM REGIMES

To solve approximatively the DDE for ISDM(CP-CLT) model, it is necessary to make a suitable revision of the corresponding potential which is known in 3D-RQM regimes as SDM Coulomb potential including a Coulomb-like tensor interaction SDM(CP-CLT) model within the framework of 3D-RQM regimes which described by the following DE as follows

$$H_D^{sc}\Psi_{nk}(r,\theta,\phi)=E_{nk}\Psi_{nk}(r,\theta,\phi) \tag{10}$$

with

$$\begin{aligned} H_D^{sc}=\widehat{\alpha}p+\widehat{\beta}\left(m(r)+S_{sc}(r)\right)\\ -i\widehat{\beta}\mathbf{r}U^{ctp}(r)+V_{sc}(r)\end{aligned} \tag{11}$$

here the vector potential $V_{sc}(r)$ and space-time scalar potential $S_{sc}(r)$ are produced from the four-vector linear momentum operator $A^{\mu}(V_{sc}(r),A=0)$ and the mass $m(r)$, respectively. The usual Dirac Hamiltonian operator H_D^{sc} for an interacted particle with the SDMC-CLT model, $\mathbf{p}=-i\hbar\nabla$ is the momentum, E_{nk} are the relativistic eigenvalues, (n,k) represent the principal and spin-orbit coupling terms. The tensor interaction $U^{ctp}(r)$ equals $\left(-\frac{H}{r},r\geq R_c\right)$, $H=\frac{Z_aZ_Ce^2}{4\pi\varepsilon_0}$, $R_c=7.78$ FM is the Coulomb radius, Z_a and Z_C denote the charges of the projectile a and the target nuclei C,[67] $\widehat{\alpha}=anti_diag\left(\sigma,\sigma\right)$, $\widehat{\beta}=diag\left(I_{2\times2},I_{2\times2}\right)$ and σ are three-vector spin matrices of three vectors. Since the SDMC-CLT model has spherical symmetry, allowing the spinor solutions $\Psi_{nk}(r,\theta,\phi)$ of the known form $\begin{pmatrix}\frac{F_{nk}^s(r)}{r}Y_{jm}^l(\theta,\phi)\\ i\frac{G_{nk}^s(r)}{r}Y_{jm}^l(\theta,\phi)\end{pmatrix}$ and $\begin{pmatrix}\frac{F_{nk}^{ps}(r)}{r}Y_{j\widetilde{m}}^{\tilde{l}}(\theta,\phi)\\ i\frac{G_{nk}^{ps}(r)}{r}Y_{j\widetilde{m}}^{\tilde{l}}(\theta,\phi)\end{pmatrix}$ for spin symmetry and pseudo-spin (p-spin) symmetry, $F_{nk}^s(r)/F_{nk}^{ps}(r)$ and $G_{nk}^s(r)/G_{nk}^{ps}(r)$ represent the upper and lower components of the Dirac spinors while $Y_{jm}^l(\theta,\phi)$ and $Y_{j\widetilde{m}}^{\tilde{l}}$ are the spin and p-spin spherical harmonics and $(m,\widetilde{m})$ are the projections on the z-axis. The upper and lower components $F_{nk}^s(r)$ and $G_{nk}^{ps}(r)$ for spin symmetry and p-spin symmetry satisfy the two uncoupled differential equations below

$$\begin{pmatrix}\frac{d^2}{dr^2}-k(k+1)r^{-2}+U_{eff}^{clp}(r)-\left(m(r)+E_{nk}-\Delta_{sc}(r)\right)\\ \left(m(r)-E_{nk}+\Sigma_{sc}(r)\right)+\frac{\left(\frac{dm(r)}{dr}-\frac{d\Delta_{sc}(r)}{dr}\right)\left(\frac{d}{dr}+\frac{k}{r}-U(r)\right)}{m(r)+E_{nk}-\Delta_{sc}(r)}\end{pmatrix}F_{nk}^s(r)=0 \tag{12}$$

and

$$\begin{pmatrix}\frac{d^2}{dr^2}-k(k-1)r^{-2}+U_{eff}^{clp}(r)-\left(m(r)+E_{nk}^{ps}-\Delta_{sc}(r)\right)\\ \left(m(r)-E_{nk}+\Sigma_{sc}(r)\right)+\frac{\left(\frac{dm(r)}{dr}-\frac{d\Sigma_{sc}(r)}{dr}\right)\left(\frac{d}{dr}-\frac{k}{r}+U(r)\right)}{m(r)+E_{nk}+\Sigma_{sc}(r)}\end{pmatrix}G_{nk}^{ps}(r)=0 \tag{13}$$

Here $U_{eff}^{clp-s/p}(r)$ equal $(\frac{2kU(r)}{r}\mp\frac{dU(r)}{dr}-U^2(r))$ which can be expressed analytically as

$$U_{eff}^{clp}=\begin{cases}-\frac{2kH}{r^2}+\frac{\left(-H-H^2\right)}{r^2}\text{ For spin-symmetry}\\ -\frac{2kH}{r^2}+\frac{\left(+H-H^2\right)}{r^2}\text{ For p-spin-symmetry}\end{cases} \tag{14}$$

While $(\Sigma_{sc}(r)$ and$\Delta_{\ sc}(r))$ are equal to $(V_{sc}(r)\pm S_{sc}(r))$, respectively and according to the data of the studied potential expressed as

$$\begin{cases}\Sigma_{sc}(r)=-\frac{C}{r}\text{ and }\frac{d\Delta_{sc}(r)}{dr}=0\Rightarrow\\ \Delta_{sc}=C_{sp}\text{: Spin symmetry}\\ \Delta_{sc}(r)=-\frac{C}{r}\text{ and }\frac{d\Sigma_{sc}(r)}{dr}=0\Rightarrow\\ \Sigma_{sc}=C_{ps}\text{: P-spin symmetry}\end{cases} \tag{15}$$

That corresponds to the p-spin symmetry, we get the following second-order Schrödinger-like equation in 3D-RQM regimes,

$$\begin{pmatrix}\frac{d^2}{dr^2}-\frac{\Lambda_k(\Lambda_k-1)+m_1(m_1+C)}{r^2}\\ +\frac{\gamma^p(m_1+C)-m_1\left(m_0+E_{nk}^p\right)}{r}-\beta^{p2}\end{pmatrix}G_{nk}^{ps}(r)=0 \tag{16}$$

with $k(k-1)$, Λ_k, γ^p and β^{p2} are equal to the corresponding values $\tilde{l}\left(\tilde{l}-1\right)$, $k+H$, $E_{nk}^{ps}-m_0-C_{ps}$ and $\left(E_{nk}^{ps}+m_0\right)\left(m_0-E_{nk}^{sp}+C_{ps}\right)$, respectively. Researchers authors for the fifth reference used the asymptotic iteration method to get the expressions for the lower component $G_{nk}^{ps}(r)$ as a function of the generalized Laguerre polynomial $L_n^{2\varepsilon_n}\left(2\beta^pr\right)$ in 3D-RQM symmetries as,

$$G_{nk}^{ps}(r)=\frac{(2\beta^p)^{\varepsilon_n^p+\frac{1}{2}}}{n!}\sqrt{\frac{(n-2\varepsilon_n)!}{n!}}r^{\varepsilon_n^p+\frac{1}{2}}\exp\left(-\beta^pr\right)L_n^{2\varepsilon_n^p}\left(2\beta^pr\right) \tag{17}$$

here

$$\varepsilon_n^p=\frac{\left(\gamma^p\left(m_1+C\right)-m_1\left(m_0+C\right)\right)}{2\beta^p}-\left(n+1/2\right)$$

That corresponds to the p-spin symmetry. The relativistic energy equation is determined by[4]

$$\begin{aligned}&\left(E_{nk}^{ps}+m_0\right)\left(m_0-E_{nk}^{ps}+C_{ps}\right)\\ &\times\left[2n+1+\sqrt{4(k+H)(k+H-1)+4m_1\left(m_1+C\right)+1}\right]^2\\ &=\left[\left(E_{nk}^{ps}+m_0-C_{ps}\right)\left(m_1+C\right)-m_1\left(m_0+E_{nk}^p\right)\right]^2\end{aligned} \tag{18}$$

To avoid repeating the solution of Eq. (12), a quick look at the relationship between the current set of parameters $(\varepsilon_n^s,\beta,\gamma)$ and the previous set $(\varepsilon_n^p,\beta^p,\gamma^p)$ reveals that the negative energy solution for pseudospin symmetry, where $V_{sc}(r)=-S_{sc}(r)$, can be obtained directly from the positive energy solution for spin symmetry by applying the parameter map[68-70]

$$F_{nk}^s(r)\leftrightarrow G_{nk}^{ps}(r),V_{sc}(r)\rightarrow-V_{sc}(r) \tag{19}$$

and

$$(\text{or }C\rightarrow-C)\text{ and }C_{ps}\leftrightarrow-C_{sp} \tag{20}$$

Apply the above equations, we found the second-order Schrödinger-like equation for spin symmetry as follows

$$\begin{pmatrix}\frac{d^2}{dr^2}-\frac{\Lambda_k(\Lambda_k-1)+m_1(m_1-C)}{r^2}\\ +\frac{\gamma(m_1-C)-m_1\left(m_0-E_{nk}^s\right)}{r}-\beta^2\end{pmatrix}F_{nk}^s(r)=0 \tag{21}$$

Allow us to obtain the following energy equation

$$\begin{aligned}&\left(m_0-E_{nk}^s\right)\left(m_0+E_{nk}^s-C_{sp}\right)\\ &\times\left[2n+1+\sqrt{4(k+H)(k+H-1)+4m_1\left(m_1-C\right)+1}\right]^2\\ &=\left[\left(m_0-E_{nk}^s+C_{sp}\right)\left(m_1-C\right)-m_1\left(m_0-E_{nk}^s\right)\right]^2\end{aligned} \tag{22}$$

The corresponding upper-spinor component $F_{nk}^s(r)$ wave function will be written as follows

$$F_{nk}^s(r)=\frac{(2\beta)^{\varepsilon_n^s+\frac{1}{2}}}{n!}\sqrt{\frac{(n-2\varepsilon_n^s)!}{n!}}r^{\varepsilon_n^s+\frac{1}{2}}\exp(-\beta r)L_n^{2\varepsilon_n^s}(2\beta r) \tag{23}$$

here

$$\left.\begin{aligned}&\varepsilon_n^s=\left(\gamma\left(m_1-C\right)-m_1\left(m_0-C\right)\right)/2\beta-\left(n+\frac{1}{2}\right)\\ &\gamma=m_0+E_{nk}^s-C_{sp}\\ &\beta^2=\left(E_{nk}^s+m_0\right)\left(m_0+E_{nk}^{sp}-C_{sp}\right)\end{aligned}\right\}$$

For the absence of the tensor interaction $H=0$, all results which we reviewed concern the p-spin symmetry and we deduced for the spin symmetry will be reduced in the work of Samer and Ramazan of the following form[5]

$$m_0^2 - E_{nk}^{p2} + C_{ps}\left(m_0 + E_{nk}^{ps}\right) = \left[\frac{q\left(m_0 - E_{nk}^{ps} + C_{ps}\right) + b\left(\frac{C_{ps}}{2} + m_0\right)}{n + \sqrt{\left(\frac{1}{2} - k\right)^2 + b(b+2q)} + \frac{1}{2}}\right]^2 \quad (24)$$

and

$$m_0^2 - E_{nk}^{s2} - C_{sp}\left(m_0 - E_{nk}^{s}\right) = \left[\frac{q\left(m_0 + E_{nk}^{s} - C_{sp}\right) + b\left(\frac{C_{sp}}{2} - m_0\right)}{n + \sqrt{\left(\frac{1}{2} + k\right)^2 + b(b-2q)} + \frac{1}{2}}\right]^2 \quad (25)$$

The lower component $G_{nk}^{s}(r)$ of spin symmetry and the upper component $F_{nk}^{ps}(r)$ of pseudospin symmetry are obtained as follows

$$\begin{pmatrix} G_{nk}^{s}(r) \\ F_{nk}^{ps}(r) \end{pmatrix} = \frac{\left(\frac{d}{dr} \pm \frac{k}{r} - U^{ctp}(r)\right)}{m(r) \pm E_{nk}^{s/p} - C_{s/ps}} \begin{pmatrix} F_{nk}^{s}(r) \\ G_{nk}^{ps}(r) \end{pmatrix}$$

3. NEW APPROXIMATE $(k,l) \neq (0,0)$ BOUND STATE SOLUTIONS OF ISDM(CP-CLTI) MODEL IN 3D-RNCS REGIMES

3.1. REVISED BOPP-SHIFT LINEAR TRANSFORMATION

Let's start this subsection by locating the deformed Dirac equation (DDE) in three-dimensional relativistic noncommutative space (3D-RNCS) symmetries using the ISDM (CP-CLTI) model. Applying the new ideas from the introduction, Eqs. (4), (5), and (9), which are contained in new relationships METNCCCRs and the concept of the Weyl-Moyal star product, allows us to achieve our goal. With the help of this data, we can rewrite the typical radial DE in Eq. (10) in the 3D-RNCS symmetries as follows:

$$\begin{pmatrix} \widehat{\alpha}p + \widehat{\beta}\left(M + S_{sc}(r)\right) - i\widehat{\beta}rU(r) \\ -\left(E_{nk} - V_{sc}(r)\right) \end{pmatrix} * \Psi_{nk}(r,\theta,\phi) = 0 \quad (26)$$

In the 3D-RNCS symmetries, allow us to rewrite the upper and lower components $F_{nk}^{s}(r)$ and $G_{nk}^{ps}(r)$ in the following second-order differential equations

$$\begin{pmatrix} \frac{d^2}{dr^2} - \frac{\Lambda_k(\Lambda_k-1)+m_1(m_1-C)}{r^2} \\ +\frac{\gamma(m_1-C)-m_1\left(m_0-E_{nk}^{s}\right)}{r} - \beta^2 \end{pmatrix} * F_{nk}^{s}(r) = 0 \quad (27)$$

and

$$\begin{pmatrix} \frac{d^2}{dr^2} - \frac{\Lambda_k(\Lambda_k-1)+m_1(m_1+C)}{r^2} \\ +\frac{\gamma^p(m_1+C)-m_1\left(m_0+E_{nk}^{p}\right)}{r} - \beta^{p2} \end{pmatrix} * G_{nk}^{ps}(r) = 0 \quad (28)$$

According to employing the Connes method[24,25] or the Seiberg and Witten map,[26] one route to discovering solutions to Eqs. (27) and (28) are the application of these methods. Specialists recognize that the Bopp-Shift linear transformation can be used to convert the star product into the typical product described in the literature. Bopp was the first to take into account the quantization rules' ability to produce pseudo-differential operators from a symbol $(x,p) \to (\widehat{x} = x - \frac{i}{2}\partial_p,\ \widehat{p} = p + \frac{i}{2}\partial_x)$ instead of ordinary correspondence $(x,p) \to (\widehat{x} = x,\ \widehat{p} = p + \frac{i}{2}\partial_x)$ respectively. For researchers, this method is often called the Bopp shifts method or the Bopp-Shift linear transformation, and this quantization method is known as Bopp quantization.[71-74] Recent years have seen a lot of success with this approach. As a result of the influence of various potentials, the nonrelativistic modified SE (MSE) has been a challenge to solve.[75-79] The MKGE,[80-87] the DDE,[88-91] and the modified Duffin-Kemmer-Petiau equation MDKPE[92,93] are just a few examples of the relativistic physics problems for which this method has been successful. The Bopp-Shift linear transformation is based on the reduction of second-order linear differential equations (MSE, MKGE, DDE, and DDKPE) with the Weyl-Moyal star product to second-order linear differential equations of Schrodinger equation, Klein-Gordon equation, Dirac equation, and Duffin-Kemmer-Petiau equation without Weyl-Moyal star product with simultaneous translation in 3D-RNCQS symmetry. Interestingly, we may reduce the mentioned equations using the Bopp-Shift linear transformation.

$$\begin{pmatrix} \frac{d^2}{dr^2} - \frac{\Lambda_k(\Lambda_k-1)+m_1(m_1-C)}{\hat{r}^2} \\ +\frac{\gamma(m_1-C)-m_1\left(m_0-E_{nk}^{s}\right)}{\hat{r}} - \beta^2 \end{pmatrix} F_{nk}^{s}(r) = 0, \quad (29)$$

and

$$\begin{pmatrix} \frac{d^2}{dr^2} - \frac{\Lambda_k(\Lambda_k-1)+m_1(m_1+C)}{\hat{r}^2} \\ +\frac{\gamma^p(m_1+C)-m_1\left(m_0+E_{nk}^{p}\right)}{\hat{r}} - \beta^{p2} \end{pmatrix} G_{nk}^{ps}(r) = 0. \quad (30)$$

Now, it is possible to obtain new MASCCCRs with usual known products in 3D-RQM symmetries, from the deformed algebraic structure of covariant canonical commutation relations with the notion of the Weyl-Moyal star product in Eqs. (5) and (6), as follows[66-69]

$$\begin{cases} \left[\widehat{x}_{\mu}^{(s,h,i)}, \widehat{p}_{\nu}^{(s,h,i)}\right] = i\hbar_{eff}\delta_{\mu\nu}\,, \\ \left[\widehat{x}_{\mu}^{(s,h,i)}, \widehat{x}_{\nu}^{(s,h,i)}\right] = i\theta_{\mu\nu}. \end{cases} \quad (31)$$

In the symmetries of 3D-RNCS, the generalized positions and momentum coordinates $(\widehat{x}_{\mu}^{(s,h,i)}, \widehat{p}_{\mu}^{(s,h,i)})$ are defined as[71-74]

$$\begin{cases} \widehat{x}_{\mu}^{(s,h,i)} = x_{\mu}^{(s,h,i)} - \sum_{\nu=1}^{3} \frac{i\theta_{\mu\nu}}{2} p_{\nu}^{(s,h,i)}, \\ \widehat{p}_{\mu}^{(s,h,i)} = p_{\mu}^{(s,h,i)}. \end{cases} \quad (32)$$

This permuted us to get the new operator $\hat{r}^2$ into two varieties in 3D-RNCQS regimes as follows

$$\hat{r}^2 = r^2 - \left(\begin{cases} \mathbf{L\Theta} & \text{for Spin symmetry} \\ \widetilde{\mathbf{L}}\mathbf{\Theta} & \text{for P-spin symmetry} \end{cases}\right) + O\left(\Theta^2\right) \quad (33)$$

while the new operators ($\hat{r}^{-2}$ and $\hat{r}^{-1}$), in the 3D-RNCS symmetries, are expressed as

$$\frac{1}{\hat{r}^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\begin{cases} \mathbf{L\Theta} & \text{for Spin symmetry} \\ \widetilde{\mathbf{L}}\mathbf{\Theta} & \text{for P-spin symmetry} \end{cases}\right) O\left(\Theta^2\right), \quad (34.1)$$

and

$$\frac{1}{\hat{r}} = \frac{1}{r} + \frac{1}{2r^3}\left(\begin{cases} \mathbf{L\Theta} & \text{for Spin symmetry} \\ \widetilde{\mathbf{L}}\mathbf{\Theta} & \text{for P-spin symmetry} \end{cases}\right) + O\left(\Theta^2\right). \quad (34.2)$$

Substituting Eqs. (34.1) and (34.2) into Eqs. (29) and (30), we get the following two like-Schrödinger equations

$$\left(\begin{matrix} \frac{d^2}{dr^2} - \frac{\Lambda_k(\Lambda_k-1)+m_1(m_1-C)}{r^2} + \\ \frac{\gamma(m_1-C)-m_1\left(m_0-E_{nk}^{s}\right)}{r} - \beta^2 - \Sigma_{sc}^{pert}(r) \end{matrix}\right) F_{nk}^{s}(r) = 0, \quad (35)$$

and

$$\left(\begin{matrix} \frac{d^2}{dr^2} - \frac{\Lambda_k(\Lambda_k-1)+m_1(m_1+C)}{r^2} + \\ \frac{\gamma^p(m_1+C)-m_1\left(m_0+E_{nk}^{p}\right)}{2r} - \beta^{p2} - \Delta_{sc}^{pert}(r) \end{matrix}\right) G_{nk}^{ps}(r) = 0. \quad (36)$$

with

$$\Sigma_{sc}^{pert}(r)=\left(\begin{array}{c}\frac{\Lambda_k(\Lambda_k+1)+m_1(m_1-C)}{r^4}\\-\frac{\gamma(m_1-C)-m_1(m_0-E_{nk}^s)}{2r^3}\end{array}\right)\mathbf{L\Theta}+O\left(\Theta^2\right),\qquad(37)$$

and

$$\Delta_{sc}^{pert}(r)=\left(\begin{array}{c}\frac{\Lambda_k(\Lambda_k-1)+m_1(m_1+C)}{r^4}\\-\frac{\gamma(m_1+C)-m_1(m_0+E_{nk}^p)}{2r^3}\end{array}\right)\widetilde{\mathbf{L}}\mathbf{\Theta}+O\left(\Theta^2\right).\qquad(38)$$

It is clear to us that the difference between the set of Eqs. (16) and (21) and the new sentence of Eqs. (35) and (36) lies in the appearance of two additive potentials ($\Sigma_{sc}^{pert}(r)$ and $\Delta_{sc}^{pert}(r)$). Additionally, these additive potentials are proportional to the infinitesimal non-commutativity couplings ($\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\mathbf{\Theta}$). From a physical point of view, this means that these additive potentials are spontaneously generated as an effect of the influence of space-space deformation. Thus, we can be considered very small compared to the original potentials ($\Sigma_{sc}(r)$ and $\Delta_{sc}(r)$), respectively. Moreover, if we use the Heaviside step function $\theta(z)$ or simply the theta function, we can rewrite the globally generated two potentials ($\Sigma_{t_{sc}}^{pert}(r)$and $\Delta_{t-sc}^{pert}(r)$) for spin and pseudospin symmetries for to upper and lower components ($F_{nk}^s(s)$, $G_{nk}^s(s)$) and ($F_{nk}^{ps}(s)$, $G_{nk}^{ps}(s)$), respectively as follows

$$\begin{aligned}\Sigma_{t_{sc}}^{pert}(r)&=\Sigma_{sc}^{pert}(r)\theta\left(E_{nc}^{sc-s}\right)-\Sigma_{sc}^{pert}(r)\theta\left(-\left|E_{nc}^{sc-s}\right|\right)\\&=\begin{cases}\Sigma_{sc}^{pert}(r)\text{ for }F_{nk}^s(r)\\-\Sigma_{sc}^{pert}(r)\text{ for }G_{nk}^s(r)\end{cases}\end{aligned}\qquad(39)$$

and

$$\begin{aligned}\Delta_{t_{sc}}^{pert}(r)&=\Delta_{sc}^{pert}(r)\theta\left(E_{nc}^{sc-ps}\right)\\&\quad-\Delta_{sc}^{pert}(r)\theta\left(-\left|E_{nc}^{sc-ps}\right|\right)\\&=\begin{cases}\Delta_{sc}^{pert}(r)\text{ for }F_{nk}^{ps}(r)\\-\Delta_{sc}^{pert}(r)\text{ for }G_{nk}^{ps}(r)\end{cases}\end{aligned}\qquad(40)$$

Here, the step function $\theta(z)$ is given by,

$$\theta(z)=\begin{cases}1\text{ for }z\geq 0\\0\text{ for }otherwise\end{cases}\qquad(41)$$

In 3D-RNCS symmetries, the SDM Coulomb potential containing a Coulomb-like tensor interaction is prolonged by including new additive potentials ($\Sigma_{sc}^{pert}(r)$ and $\Delta_{sc}^{pert}(r)$) that expressed to the radial terms ($\frac{1}{r^3}$ and $\frac{1}{r^4}$) which are combined with new couplings ($\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\mathbf{\Theta}$) to give the improved SDM Coulomb potential including a Coulomb-like tensor interaction. The two globally generated potentials ($\Sigma_{t_{sc}}^{pert}(r)$ and $\Delta_{t-sc}^{pert}(r)$) describes the physical interaction between the SDM Coulomb potential and the physical properties that correspond to spin and pseudospin symmetries ($\mathbf{L}$,$\widetilde{\mathbf{L}}$) with the influence of space-space deformation which is characterized by non-commutativity vector $\mathbf{\Theta}$. The physical behavior of both two perturbed effective potentials ($\Sigma_{sc}^{pert}(r)$, $\Delta_{sc}^{pert}(r)$) quite similar to their original ($\Sigma_{sc}(r)$, $\Delta_{sc}(r)$) in terms of their dependence on the two infinitesimal couplings ($\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\mathbf{\Theta}$). This permuted us to consider the new additive-generated potentials ($\Sigma_{sc}^{pert}(r)$ and $\Delta_{sc}^{pert}(r)$) as perturbation potentials compared with the main potentials ($\Sigma_{sc}(r)$ and $\Delta_{sc}(r)$) which are also known with the parent potential operator in the symmetries of DDE, that is, the two inequalities ($\Sigma_{sc}^{pert}(r)<<\Sigma_{sc}(r)$ and $\Delta_{sc}^{pert}(r)<<\Delta_{sc}(r)$) have become achieved. All this physical information confirms that the application of the time-independent perturbation theory is the physical tool that guarantees the treatment of the studied issue with high efficiency and accuracy to find various corrections for determining the energy level of the generalized $\left(n,l,\tilde{l},m,\tilde{m},s,\tilde{s}\right)^{th}$ excited states.

3.2. THE RELATIVISTIC EXPECTATION VALUES UNDER THE ISDM (CP-CLTI) MODEL IN THE 3D-RNCS SYMMETRIES FOR SPIN SYMMETRY

In this subsection, we want to use the time-independent perturbation theory (SPT), in the context of 3D-RNCQS symmetries, and that's to find both

$$M_{1(nlms)}^{sp-sc}\equiv\left\langle\frac{1}{r^3}\right\rangle_{(nlms)}^{sp-sc}$$

and

$$M_{2(nlms)}^{sp-sc}\equiv\left\langle\frac{1}{r^4}\right\rangle_{(nlms)}^{sp-sc}$$

That corresponds to the spin symmetry taking into consideration the unperturbed form of the upper component $F_{nk}^s(r)$ that we have seen previously in Eq. (23) (second chapter). Profound calculation gives $M_{1(nlms)}^{sp-sc}$ and $M_{2(nlms)}^{sp-sc}$ as follows

$$\begin{aligned}&M_{1(nlms)}^{sp-sc}\\&=\frac{(2\beta)^{2\varepsilon_n^s+1}}{n!^3}(n-2\varepsilon_n^s)!\\&\quad\times\int_0^{+\infty}r^{2\varepsilon_n^s-1-1}\exp(-2\beta r)\left[L_n^{2\varepsilon_n^s}(2\beta r)\right]^2dr,\end{aligned}\qquad(42.1)$$

and

$$\begin{aligned}&M_{2(nlms)}^{sp-sc}\\&=\frac{(2\beta)^{2\varepsilon_n^s+1}}{n!^3}(n-2\varepsilon_n^s)!\\&\quad\times\int_0^{+\infty}r^{2\varepsilon_n^s-2-1}\exp(-2\beta r)\left[L_n^{2\varepsilon_n^s}(2\beta r)\right]^2dr.\end{aligned}\qquad(42.2)$$

The abbreviations $\langle R\rangle_{(nlms)}^{sp-sc}=\langle n,l,m\ \lfloor R\rfloor\ n,l,m\rangle$ is applied to avoid the extra burden of writing, here $R=\left(\frac{1}{r^3}\text{ or }\frac{1}{r^4}\right)$. Additionally, we have applied the property of spherical harmonics, which has the form

$$\iint Y_m^l(\theta,\phi)Y_{m'}^{l'}(\theta,\phi)d\Omega=\delta_{ll'}\delta_{mm'}\qquad(43)$$

Here $d\Omega$ is a differential solid angle. Comparison of Eqs. (42.1) and (42.2) with the integral of form[94]

$$\begin{aligned}&\int_0^{+\infty}t^{\eta-1}\exp(-\gamma t)L_m^\lambda(\gamma t)L_n^\beta(\gamma t)dt\\&=\frac{\gamma^{-\eta}\Gamma(n-\eta+\beta+1)\Gamma(m+\lambda+1)}{m!n!\Gamma(1-\eta+\beta)\Gamma(\lambda+1)}\\&\quad\times{}_3F_2(-m,\eta,\eta-\beta;-n+\eta,\lambda+1,1)\end{aligned}\qquad(44)$$

her ${}_3F_2(-m,\eta,\eta-\beta;-n+\eta,\lambda+1,1)$ is a special case from the generalized hypergeometric function ${}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta^1,\ldots,\beta^q,1\right)$ when ($p=3$ and $q=2$) while $\Gamma(x)$ just the ordinary Gamma function. After straightforward calculations we find

$$\begin{aligned}&M_{1(nlms)}^{sp-sc}\\&=\frac{(2\beta)^2}{n!^4}(n-2\varepsilon_n^s)!\frac{(n+1)\Gamma\left(n+2\varepsilon_n^s+1\right)}{\Gamma\left(2\varepsilon_n^s+1\right)}\\&\quad\times{}_3F_2\left(-n,2\varepsilon_n^s-1,-1;-n+2\varepsilon_n^s-1,2\varepsilon_n^s+1,1\right),\end{aligned}\qquad(45.1)$$

and

$$\begin{aligned}&M_{2(nlms)}^{sp-sc}\\&=\frac{(2\beta)^3}{n!^4}(n-2\varepsilon_n^s)!\frac{(n+2)(n+1)\Gamma\left(n+2\varepsilon_n^s+1\right)}{2\Gamma\left(2\varepsilon_n^s+1\right)}\\&\quad\times{}_3F_2\left(-n,2\varepsilon_n^s-2,-2;-n+2\varepsilon_n^s-2,2\varepsilon_n^s+1,1\right).\end{aligned}\qquad(45.2)$$

we have used the well-known property $\Gamma(n+1)=n!$.

3.3. THE NEW RELATIVISTIC EXPECTATION VALUES UNDER THE ISDM(CP-CLTI) MODEL IN THE 3D-RNCS SYMMETRIES FOR THE PSEUDOSPIN SYMMETRY

In this subsection, we want to use the SPT; in the case of 3D-RNCQS symmetries, we obtain the relativistic expectation values of the form

$$M^{ps-sc}_{1\left(n\tilde{l}\tilde{m}\tilde{s}\right)}\equiv\left\langle\frac{1}{r^3}\right\rangle^{ps-sc}_{\left(n\tilde{l}\tilde{m}\tilde{s}\right)}\text{and}M^{ps-sc}_{2\left(n\tilde{l}\tilde{m}\tilde{s}\right)}\equiv\left\langle\frac{1}{r^4}\right\rangle^{ps-sc}_{\left(n\tilde{l}\tilde{m}\tilde{s}\right)}$$

That corresponds to the case of the pseudo-spin regime with tensor interaction taking into account the unperturbed wave function which we have seen previously in Eq. (17). By observing the equations expressing each of the upper and lower components ($F^s_{nk}(r)$ and $G^{ps}_{nk}(r)$) shown in Eqs. (23) and (17), we note that there is a possibility of moving from the unperturbed upper component $F^s_{nk}(r)$to the other lower component $G^{ps}_{nk}(r)$ by making appropriate transfers for this purpose

$$\beta\Leftrightarrow\beta^p\text{ and }\varepsilon^s_n\Leftrightarrow\varepsilon^p_n \tag{46}$$

This permuted us to get the new physical relativistic expectation values for the pseudospin symmetry from Eqs. (45.1) and (45.2) without re-calculation,

$$\begin{aligned}&M^{sp-sc}_{1\left(n\tilde{l}\tilde{m}\tilde{s}\right)}\\&=\frac{\left(2\beta^p\right)^2(n-2\varepsilon^p_n)!}{n!^4}\frac{(n+1)\Gamma\left(n+2\varepsilon^s_n+1\right)}{\Gamma\left(2\varepsilon^s_n+1\right)}\\&\times{}_3F_2\left(-n,2\varepsilon^s_n-1,-1;-n+2\varepsilon^s_n-1,2\varepsilon^s_n+1,1\right),\end{aligned} \tag{47.1}$$

and

$$\begin{aligned}&M^{sp-sc}_{2\left(n\tilde{l}\tilde{m}\tilde{s}\right)}\\&=\frac{\left(2\beta^p\right)^3(n-2\varepsilon^p_n)!}{n!^4}\frac{(n+2)(n+1)\Gamma\left(n+2\varepsilon^p_n+1\right)}{2\Gamma\left(2\varepsilon^p_n+1\right)}\\&\times{}_3F_2\left(-n,2\varepsilon^p_n-2,-2;-n+2\varepsilon^p_n-2,2\varepsilon^p_n+1,1\right).\end{aligned} \tag{47.2}$$

3.4. THE NEW CORRECTED ENERGY FOR ISDM(CP-CLTI) MODEL IN 3D-RNCS SYMMETRIES

We aim through this subsection to achieve the contribution produced from deformed space-space properties based on our proper strategy, which we have successfully used in our previous reaches and which we try to develop in every new reach. We can say, that the total new relativistic energy in the perspective of deformation Dirac theory is generated under the ISDM (CP-CLTI) model. This is a direct consequence of a major contribution to relativistic energy known in the 3D-RQM symmetries under the SDMCCLT model in usual Dirac theory, which we paved for through a quick look for the spin and (pseudospin)-symmetry in Eqs. (18) and (22) (see second chapter). The new physical contribution is generated from the effect of space-space deformation, which can be evaluated through several contributions; we present the important fundamental physics contributions in three forms that are vital to the outputs of quantum mechanics. The first contribution is induced from the effect of the perturbed spin-orbit effective potentials ($\Sigma^{pert}_{sc}(r)$ and $\Delta^{pert}_{sc}(r)$) corresponds to spin symmetry and pseudospin symmetry. These perturbed effective potentials are obtained by replacing the coupling of the angular momentum ($\mathbf{L}$ and $\widetilde{\mathbf{L}}$) operators and the NC vector $\mathbf{\Theta}$ with the new equivalent couplings ($\Theta\mathbf{LS}$ and $\Theta\widetilde{\mathbf{LS}}$) for spin-symmetry and pseudospin-symmetry, respectively (with $\Theta^2=\Theta^2_{12}+\Theta^2_{23}+\Theta^2_{13}$). This is because the NC vector $\mathbf{\Theta}$ is an arbitrary value, which allows us to deal with its value according to the physical need. We have oriented the two spin-s and spin- of the fermionic particles to become parallel to the vector that interacted under the ISDM (CP-CLTI) model. Additionally, it is appropriate to change the new spin-orbit couplings ($\Theta\mathbf{LS}$ and $\Theta\widetilde{\mathbf{LS}}$) with the new corresponding physical form ($(\frac{\Theta}{2})\mathbf{G^2}$and $(\frac{\Theta}{2})\widetilde{\mathbf{G}}^{\mathbf{2}}$), as follows

$$\left.\begin{aligned}\mathbf{G}^2&=\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2\\\widetilde{\mathbf{G}}^2&=\mathbf{J}^2-\tilde{\mathbf{L}}^2-\tilde{\mathbf{S}}^2\end{aligned}\right\}$$

That corresponds a spin and pseudospin regimes, respectively. Additionally, in 3D-RQM symmetry, the operators (H^{sc}_{rnc}, J^2, L^2, S^2 and J_z) form a complete set of conserved physics quantities and the eigenvalues of the operators $G^2$and $\widetilde{G}^2$are equal to the new physical values

$$\begin{cases}2F(j,l,s)=j(j+1)-l(l+1)-s(s+1)\\\quad\text{with: }|l-s|\leq j\leq+|l+s|\\2F\left(j,\tilde{l},\tilde{s}\right)=j(j+1)-\tilde{l}\left(\tilde{l}-1\right)-\tilde{s}\left(\tilde{s}+1\right)\\\quad\text{with: }\left|\tilde{l}-\tilde{s}\right|\leq j\leq+\left|\tilde{l}+\tilde{s}\right|\end{cases}$$

That corresponds to the spin and pseudospin regimes, respectively. As a direct consequence, the partially corrected energies $\Delta E^{so-sp}_{sc}\left(n,C,m_0,m_1,H,\Theta,j,l,s\right)\equiv\Delta E^{so-sp}_{sc}$and $\Delta E^{so-ps}_{sc}\left(n,C,m_0,m_1,H,\Theta,j,\tilde{l},\tilde{s}\right)\equiv\Delta E^{so-ps}_{sc}$ due to the perturbed effective potentials ($\Sigma^{pert}_{sc}(r)$ and $\Delta^{pert}_{sc}(r)$) produced for the $\left(n,l,\tilde{l},m,\widetilde{m},s,\tilde{s}\right)^{th}$ excited state, in the context of deformation Dirac theory, as follows

$$\begin{gathered}\Delta E^{so-sp}_{sc}=\Theta F(j,l,s)\left\langle X\right\rangle^{sc}_{(nlms)}\left(n,C,m_0,m_1,H\right)\\\text{For-}F^s(r)\text{Spin symmetry,}\end{gathered} \tag{48.1}$$

and

$$\begin{gathered}\Delta E^{so-ps}_{sc}=\Theta F\left(j,\tilde{l},\tilde{s}\right)\left\langle\widetilde{X}\right\rangle^{sc}_{\left(n\tilde{l}\tilde{m}\tilde{s}\right)}\left(n,C,m_0,m_1,H\right)\\\text{For-}G^{ps}(r)\text{ (P-spin symmetry).}\end{gathered} \tag{48.2}$$

The two global physical expectation values $\left\langle X\right\rangle^{sc}_{(nlms)}\left(n,C,m_0,m_1,H\right)$ and $\left\langle\widetilde{X}\right\rangle^{sc}_{\left(n\tilde{l}\tilde{m}\tilde{s}\right)}\left(n,C,m_0,m_1,H\right)$ for spin and pseudospin regimes, respectively, which were produced under the ISDM (CP-CLTI) model, are given from the new expressions

$$\begin{aligned}&\left\langle X\right\rangle^{sc}_{(nlms)}\left(n,C,m_0,m_1,H\right)\\&=\left(\Lambda_k\left(\Lambda_k+1\right)+m_1\left(m_1-C\right)\right)\left\langle\frac{1}{r^4}\right\rangle^{sp-sc}_{(nlms)}\\&-\frac{1}{2}\left(\gamma\left(m_1-C\right)-m_1\left(m_0-E^s_{nk}\right)\right)\left\langle\frac{1}{r^3}\right\rangle^{sp-sc}_{(nlms)},\end{aligned} \tag{49.1}$$

and

$$\begin{aligned}&\left\langle\widetilde{X}\right\rangle^{sc}_{\left(n\tilde{l}\tilde{m}\tilde{s}\right)}\left(n,C,m_0,m_1,H\right)\\&=\left(\Lambda_k\left(\Lambda_k-1\right)+m_1\left(m_1+C\right)\right)\left\langle\frac{1}{r^4}\right\rangle^{ps-sc}_{\left(n\tilde{l}\tilde{m}\tilde{s}\right)}\\&-\frac{1}{2}\left(\gamma\left(m_1+C\right)-m_1\left(m_0+E^p_{nk}\right)\right)\left\langle\frac{1}{r^3}\right\rangle^{ps-sc}_{\left(n\tilde{l}\tilde{m}\tilde{s}\right)}.\end{aligned} \tag{49.2}$$

Now we will discuss the second major contribution represented by the magnetic effect of the perturbative effective potentials ($\Sigma^{pert}_{sc}(r)$and $\Delta^{pert}_{sc}(r)$) under the effect of ISDM(CP-CLTI) model in the 3D-RNCS regimed. These effective potentials are obtained when we replace both ($\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\mathbf{\Theta}$) with new corresponding values ($\tau\aleph L_z$ and $\tau\aleph\widetilde{L}_z$), respectively. We just replaced Θ_{13} by $\tau\aleph$. with ($\aleph$and τ) present the intensity of the magnetic field induced by the effect of the influence of space-space deformation. The new infinitesimal noncommutativity parameter so that the

physical unit of the original noncommutativity parameter (length)[95] is the same unit of $\tau\aleph$, ($[\Theta_{13}]=[\tau][\aleph]$), We'll need to apply the following relations to achieve this

$$\begin{cases} \left\langle n^{'},l^{'},m^{'} \ \| L_z \| \ n,l,m \right\rangle = m\delta_{m^{'}m}\delta_{l^{'}l}\delta_{n^{'}n} \\ \quad \text{with: } \ m \in [-l,+l] \\ \left\langle n^{'},\tilde{l}^{\prime},\widetilde{m}^{\prime} \left\| \widetilde{L}_z \right\| \ n,\tilde{l},\widetilde{m} \right\rangle = m^{p}\delta_{\widetilde{m}\widetilde{m}^{\prime}}\delta_{\tilde{l}^{\prime}\tilde{l}}\delta_{n^{\prime}n} \\ \quad \text{with: } \widetilde{m} \in \left[-\tilde{l},+\tilde{l}\right] \end{cases}$$

That corresponds to the spin and pseudospin-regimes, respectively. All of these physical considerations permuted us to get the new energy shift $\Delta E_{sc}^{mg-s}\ (n,C,m_0,m_1\ ,\tau,m)$ and $\Delta E_{sc}^{mg-ps}\ (n,C,m_0,m_1,\tau,\widetilde{m})$ due to the perturbed Zeeman effect created by the influence of the ISDM(CP-CLTI) model for the $\left(n,l,\tilde{l},m,\widetilde{m},s,\tilde{s}\right)^{th}$ excited state in deformation Dirac theory symmetries as follows

$$\begin{aligned} &\Delta E_{sc}^{mg-sp}\left(n,C,m_0,m_1,H,\tau,m\right) \\ &\quad = \tau\aleph\langle X\rangle_{(nlms)}^{sc} m : \text{For-upper-component } F^{s}(r), \end{aligned} \tag{50.1}$$

and

$$\begin{aligned} &\Delta E_{sc}^{mg-ps}\left(n,C,m_0,m_1,H,\tau,\widetilde{m}\right) \\ &\quad = \tau\aleph\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}\widetilde{m} : \text{For-lower-component } G^{ps}(r). \end{aligned} \tag{50.2}$$

After we have accomplished the previous two physical cases, it is useful to address another case that we think is useful for our case. This new physical phenomenon is generated automatically from the effect of perturbed effective potentials ($\Sigma_{sc}^{pert}(r)$ and $\Delta_{sc}^{pert}(r)$) which we have seen in Eqs. (37) and (38), respectively. We consider the fermionic particles undergoing rotation with angular velocity Ω. The features of this subjective phenomenon are determined by replacing the arbitrary vector $\mathbf{\Theta}$ with a new vector $\chi\mathbf{\Omega}$. Allowing us to replace the two couplings ($\mathbf{L\Theta}$ and $\widetilde{\mathbf{L}}\mathbf{\Theta}$) with ($\chi\mathbf{L\Omega}$ and $\chi\widetilde{\mathbf{L}}\mathbf{\Omega}$), respectively, as follows

$$\left(\mathbf{L\Theta}\ and\ \widetilde{\mathbf{L}}\mathbf{\Theta}\right)\rightarrow\chi\left(\mathbf{L\Omega}\ \ and\ \widetilde{\mathbf{L}}\mathbf{\Omega}\right) \tag{51}$$

here X is a new infinitesimal real proportional parameter. The new effective generated perturbed potentials ($\Sigma_{pert}^{sc-rot}(s)$ and $\Delta_{pert}^{sc-rot}(s)$), due to the rotational movements of the fermionic particles, can be expressed as follows

$$\begin{aligned} \Sigma_{pert}^{sc-rot}(r) = \chi &\begin{pmatrix} \frac{\Lambda_k(\Lambda_k+1)+m_1(m_1-C)}{r^4} \\ -\frac{\gamma(m_1-C)-m_1\left(m_0-E_{nk}^{s}\right)}{2r^3} \end{pmatrix}\mathbf{L\Omega} \\ &+O\left(\chi^2\right), \end{aligned} \tag{52.1}$$

and

$$\begin{aligned} \Delta_{pert}^{sc-rot}(r) = \chi &\begin{pmatrix} \frac{\Lambda_k(\Lambda_k-1)+m_1(m_1+C)}{r^4} \\ -\frac{\gamma(m_1+C)-m_1\left(m_0+E_{nk}^{p}\right)}{2r^3} \end{pmatrix}\widetilde{\mathbf{L}}\mathbf{\Omega} \\ &+O\left(\chi^2\right). \end{aligned} \tag{52.2}$$

We have oriented the rotational velocity to become parallel to the (Oz) axis ($\Omega=\Omega e_z$) to simplify the calculations. This, of course, does not change the physical characteristics of the problem as much as it simplifies the calculations. Thus, the two rotational movements $\mathbf{L\Omega}$ and $\widetilde{\mathbf{L}}\mathbf{\Omega}$ can be transformed into a new physical simplified form as follows

$$\begin{pmatrix}\mathbf{L\Omega}\\ \widetilde{\mathbf{L}}\mathbf{\Omega}\end{pmatrix}\rightarrow\chi\begin{pmatrix}L_z\\ \widetilde{L}_z\end{pmatrix}\Omega \tag{53}$$

All of these physical considerations permuted us to discover the new corrected energy $\Delta E_{sc}^{rot-sp}\ (n,C,m_0,m_1,\chi,m)$ and $\Delta E_{sc}^{rot-ps}\ (n,C,m_0,m_1,\chi,\widetilde{m})$ due to the perturbed two effective potentials ($\Sigma_{pert}^{sc-rot}(r)$ and $\Delta_{pert}^{sc-rot}(r)$) which are generated automatically with the effect of the improved spatially dependent mass Coulomb potential including a Coulomb-like tensor interaction for the $\left(n,l,\tilde{l},m,\widetilde{m},s,\tilde{s}\right)^{th}$ excited state in MDT symmetries as follows

$$\begin{aligned} &\left(\Delta E_{sc}^{rot-sp},\Delta E_{sc}^{rot-ps}\right) \\ &\quad = \chi\left(\langle X\rangle_{(nlms)}^{sc}(n,C,m_0,m_1)m,\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,C,m_0,m_1)\widetilde{m}\right)\Omega \end{aligned} \tag{54}$$

It is useful to point out that in previous studies, the researcher's authors for reference[96] investigated rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two- and three-dimensional space at zero temperature. In this context, the rotational term was added manually to the expression of the Hamiltonian operator. Whereas in our current study, the two perturbed rotation operators ($\Sigma_{pert}^{sc-rot}(r)\mathbf{L\Omega}$, $\Delta_{pert}^{sc-rot}(r)\widetilde{\mathbf{L}}\mathbf{\Omega}$) generated automatically as a consequence of the influence of space-space deformation under the improved spatially dependent mass Coulomb potential including a Coulomb-like tensor interaction model. For a fermionic particle and ant-particle (negative energy), the eigenvalues of the operations (G^2 and $\widetilde{G}^2$) with spin-½ are expressed by the following values

$$\left.\begin{aligned} 2F(j,l,s) &= j(j+1)-l(l+1)-3/4 \\ 2F\left(j,\tilde{l},\tilde{s}\right) &= j(j+1)-\tilde{l}\left(\tilde{l}-1\right)-3/4 \end{aligned}\right\}$$

Respectively. Thus, the possible physical values for j are ($l\pm\frac{1}{2}$ and $\tilde{l}\pm$ ½) for spin symmetry $F(j,l,s)$ and pseudospin symmetry $F\left(j,\tilde{l},\tilde{s}\right)$, are as follows

$$\begin{aligned} &F^{f}(j=l\pm½,s=½) \\ &\quad = \frac{1}{2}\begin{cases} l \text{ for } j=l+½: \text{ For up-polarity} \\ -(l+1) \text{ for } j=\ l-½: \text{ For Down-polarity} \end{cases} \end{aligned} \tag{55}$$

and

$$\begin{aligned} &F^{f}\left(j=\tilde{l}\pm½,\tilde{s}=½\right) \\ &\quad = \frac{1}{2}\begin{cases} \tilde{l} \text{ with } j=\tilde{l}+½: \text{For up-polarity} \\ -\left(\tilde{l}+1\right) \text{ with } j=\ \tilde{l}-½: \text{For Down-polarity} \end{cases} \end{aligned} \tag{56}$$

In 3D-RNCS symmetries, the total relativistic energy $E_{nc}^{sp}(n,C,m_0,m_1,H,\Theta,\tau,\chi,j,l,s,m)$ and $E_{nc}^{ps}(n,C,m_0,m_1,H,\Theta,\tau,\ \chi,j,l,\tilde{l},\tilde{s},\widetilde{m})$ for the case of spin-½, for example, H-atoms (He^{+}, Li^{+2}, Be^{3+}) with improved SDM Coulomb potential including a Coulomb-like tensor interaction model, corresponding to the generalized $\left(n,l,\tilde{l},m,\widetilde{m},s,\tilde{s}\right)^{th}$ excited states are expressed as

$$\begin{aligned} &E_{nc}^{sp}\left(n,C,m_0,m_1,H,\Theta,\tau,\chi,j,l,s,m\right) \\ &\quad = E_{nk}^{s}+\langle X\rangle_{(nlms)}^{sc}\left(n,C,m_0,m_1\right) \\ &\qquad \times\left((\tau\aleph+\chi\Omega)m\frac{\Theta}{2}\begin{cases} l \text{ for } j=l+½ \\ -(l+1) \text{ for } j=\ l-½ \end{cases}\right), \end{aligned} \tag{57}$$

And

$$\begin{aligned} &E_{nc}^{ps}\left(n,C,m_0,m_1,H,\Theta,\tau,\chi,j,\tilde{l},\widetilde{m},\tilde{s}\right) \\ &\quad = E_{nk}^{ps}+\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}\left(n,C,m_0,m_1\right) \\ &\qquad \times\left((\tau\aleph+\chi\Omega)\widetilde{m}+\frac{\Theta}{2}\begin{cases} \tilde{l} \text{ for } j=\tilde{l}+½ \\ -\left(\tilde{l}+1\right) \text{ for } j=\tilde{l}-½ \end{cases}\right). \end{aligned} \tag{58}$$

Here (E_{nk}^{s} and E_{nk}^{ps}) are just the ordinary relativistic energies under SDM Coulomb potential including a Coulomb-like tensor interaction model obtained from equations of energy in Eqs. (22) and (18) in the context of 3D-RQM regimes. These results describe spin and pseudospin new

energies in DDE for atoms with one electron such as H-atoms (He^{+}, Li^{+2}, Be^{3+}). For fermionic particles or anti-particles with spin-$(s/\tilde{s})$, we replace $F^{f}(j,l,s)$ and $F^{f}\left(j,\tilde{l},\tilde{s}\right)$ by the generalized two previous values $F(j,l,s)$ and$F\left(j,\tilde{l},\tilde{s}\right)$. Allow us to obtain the total relativistic energies $E_{nc}^{sp}\ (n,C,m_0,m_1,H,\Theta,\tau,\chi,j,l,s,m)$ and E_{nc}^{ps} $(n,C,m_0,m_1,\Theta,\tau,\chi,j,\tilde{l},\tilde{s},\widetilde{m})$ with improved SDM Coulomb potential including a Coulomb-like tensor interaction model, corresponding to the generalized $\left(n,l,\tilde{l},m,\widetilde{m},s,\tilde{s}\right)^{th}$ excited states are expressed as

$$\begin{aligned} &E_{nc}^{sp}\left(n,C,m_0,m_1,H,\Theta,\tau,\chi,j,l,s,m\right)\\ &=E_{nk}^{s}+\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,C,m_0,m_1\right)\\ &\times\left[(\tau\aleph+\chi\Omega)m+F^{f}(j,l,s)\right], \end{aligned} \tag{59}$$

and

$$\begin{aligned} &E_{nc}^{ps}\left(n,C,m_0,m_1,H,\Theta,\tau,\chi,j,\tilde{l},\tilde{s},\widetilde{m}\right)\\ &=E_{nk}^{ps}+\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}\left(n,C,m_0,m_1\right)\\ &\times\left[(\tau\aleph+\chi\Omega)\widetilde{m}+\frac{\Theta}{2}F^{f}\left(j,\tilde{l},\tilde{s}\right)\right]. \end{aligned} \tag{60}$$

We can now generalize our obtained energies (E_{g-nc}^{sc-s} and E_{g-nc}^{sc-p}) under the improved SDM Coulomb potential including a Coulomb-like tensor interaction model produced with the globally induced two potentials $\Sigma_{t_{sc}}^{pert}(r)$ and $\Delta_{t-sc}^{pert}(r)$ for spin and pseudospin symmetries corresponding to the upper component (Uc) and lower component (Lc) ($F_{nk}^{s}(s)$ and $G_{nk}^{s}(s)$) and ($F_{nk}^{ps}(s)$and $G_{nk}^{ps}(s)$), respectively as

$$\begin{aligned} E_{g-nc}^{sc-s}&=E_{nc}^{sc-s}\theta\left(\left|E_{nc}^{sc-s}\right|\right)\\ &-E_{nc}^{sc-s}\theta\left(-\left|E_{nc}^{sc-s}\right|\right)\\ &=\begin{cases}E_{nc}^{sc-s}:\text{For Uc }F_{nk}^{s}(s)\ \text{(spin symmetry)}\\ -E_{nc}^{sc-s}:\text{For Lc }G_{nk}^{s}(s)\ \text{(spin symmetry)}\end{cases}, \end{aligned} \tag{61}$$

and

$$\begin{aligned} E_{g-nc}^{sc-ps}&=E_{nc}^{sc-ps}\theta\left(\left|E_{nc}^{sc-ps}\right|\right)\\ &-E_{nc}^{sc-ps}\theta\left(-\left|E_{nc}^{sc-ps}\right|\right)\\ &=\begin{cases}E_{nc}^{sc-ps}:\text{For Uc }F_{nk}^{ps}(s)\ \text{(p-spin symmetry)}\\ -E_{nc}^{sc-ps}:\text{For Lc }G_{nk}^{s}(s)\ \text{(p-spin symmetry)}\end{cases}. \end{aligned} \tag{62}$$

As far as we know, the new energy spectra given in Eqs. (59) and (60) for the $(k,l)\neq(0,0)$ bound state solutions for the ISDM(CP-CLTI) model are new and no previous researcher has obtained them.

3.5. STUDY SOME USEFUL RELATIVISTIC APPLICATIONS IN 3D-RNCS SYMMETRIES

After studying the relativistic solutions of the ISDM (CP-CLTI) model, we examine some important cases derived from Eqs. (59) and (60) in the context of RQM symmetries which are treated within the context of 3D-RQM symmetries in the main Ref.,[5] and we are now in the process of restudied taking into account the effect of non-commutativity influences

By assuming, in Eq. (57), the parameters $V_{sc}(r)=S_{sc}(r)$ and $H=0$, i.e., $q=q_v$ or $C_{sp}=0$ one finds directly

$$\begin{aligned} &E_{nc}^{p}\left(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m\right)\\ &=\frac{q(b-q)+B_{nk}^{s}\sqrt{B_{nk}^{s2}-b(b-2q)}}{q^2+B_{nk}^{s2}}m_0\\ &+\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,q,m_0,m_1\right)\\ &\times\left((\tau\aleph+\chi\Omega)m+\frac{\Theta}{2}\begin{cases}l\text{ for }j=l+½\\ -(l+1)\text{ for }j=\ l-½\end{cases}\right), \end{aligned} \tag{63}$$

and

$$\begin{aligned} &E_{nc}^{ap}\left(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m\right)\\ &=\frac{q(b-q)-B_{nk}\sqrt{B_{nk}^{2}-b(b-2q)}}{q^2+B_{nk}^{s2}}m_0\\ &+\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,q,m_0,m_1\right)\\ &\times\left((\tau\aleph+\chi\Omega)m+\frac{\Theta}{2}\begin{cases}l\text{ for }j=l+½\\ -(l+1)\text{ for }j=\ l-½\end{cases}\right). \end{aligned} \tag{64}$$

with

$$B_{nk}^{s}=n+\frac{1}{2}+\sqrt{(k-½)^2+b(b-2q)}$$

The first two parts in the right hand said of Eqs. (63) and (64) describe the usual relativistic energy of fermionic particles and fermionic anti-particles within the context of 3D-RQM symmetries. The rest terms present the effect of space-space deformation on these main energies in 3D-RQM regimes. However, $\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,q,m_0,m_1\right)$ it can be obtained by applying the limit

$$\begin{aligned} &\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,q,m_0,m_1\right)\\ &=\underset{(C,H)\rightarrow(q,0)}{Lim}\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,C,m_0,m_1,H\right) \end{aligned} \tag{65}$$

If we consider the case when $V_{sc}(r)=S_{sc}(r)$ and $H=0$, i.e., $q=q_v$ or $C_{sp}=0$ and $m_1=0$, then Eq. (57) simplified to the following expressions $E_{nc}^{p}\left(n,q,m_0,\Theta,\tau,\chi,j,l,s,m\right)$ and $E_{nc}^{ap}\left(n,q,m_0,\Theta,\tau,\chi,j,l,s,m\right)$ as follows

$$\begin{aligned} &E_{nc}^{p}\left(n,q,m_0,\Theta,\tau,\chi,j,l,s,m\right)\\ &=\frac{(n+k+1)^2-q^2}{(n+k+1)^2-q^2}m_0\\ &+\left\langle Z\right\rangle_{(nlms)}^{sc}\left(n,q,m_0\right)\\ &\times\left((\tau\aleph+\chi\Omega)m+\frac{\Theta}{2}\begin{cases}l\text{ for }j=l+½\\ -(l+1)\text{ for }j=\ l-½\end{cases}\right). \end{aligned} \tag{66}$$

and

$$\begin{aligned} &E_{nc}^{ap}\left(n,q,m_0,\Theta,\tau,\chi,j,l,s,m\right)\\ &=-m_0+\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,q,m_0\right)\\ &\times\left((\tau\aleph+\chi\Omega)m+\frac{\Theta}{2}\begin{cases}l\text{ for }j=l+½\\ -(l+1)\text{ for }j=\ l-½\end{cases}\right). \end{aligned} \tag{67}$$

The first two parts in the right hand said of Eqs. (66) and (67) describe the ordinary relativistic energy of particle and anti-particle within the context of 3D-RQM regimes known in the literature. The remaining terms describe the influence of space-space deformation on these main energies. While the new expectation value $\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,q,m_0\right)$ can be obtained by applying the limit

$$\begin{aligned} &\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,q,m_0\right)\\ &=\underset{(C,H,m_1)\rightarrow(q,0,0)}{Lim}\left\langle X\right\rangle_{(nlms)}^{sc}\left(n,C,m_0,m_1,H\right) \end{aligned} \tag{68}$$

For the s-wave which corresponds $l=0$ or $k=-1$ and $m=0$, Eq. (66) reduces to

$$\begin{aligned} E_{nc}^{p}\left(n,q,m_0,\Theta,\tau,\chi,j,l,s,m\right)&=\frac{(n+1)^2-q^2}{(n+1)^2-q^2}m_0\\ &+\frac{\Theta}{2}\left\langle X\right\rangle_{(nlms)}^{sc}\\ &\times\left(n,q,m_0\right)\begin{cases}0\text{ for }j=½\\ -1\text{ for }j=\ -½\end{cases} \end{aligned} \tag{69}$$

The first part in the right hand said of Eq. (69) describes the ordinary relativistic energy of particles for the *s*-wave within the framework of relativistic QM known in the liter-

ature. The remaining terms present the influence of space-space deformation on these main energies in 3D-RQM regimes.

By assuming, in Eq. (58), the parameters $V_{sc}(r)=-S_{sc}(r)$ and $H=0$, i.e., $q_s=-q_v$ or $C_{ps}=0$ one finds directly

$$\begin{aligned} &E_{nc}^{p}\left(n,q,m_0,m_1,\Theta,\tau,\chi,j,\tilde{l},\tilde{s},\widetilde{m}\right)\\ &=\frac{-q(b+q)+B_{nk}^{p}\sqrt{B_{nk}^{p2}-b(b+2q)}}{q^2+B_{nk}^{p2}}+\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,q,m_0,m_1)\\ &\times\left((\tau\aleph+\chi\Omega)\widetilde{m}+\frac{\Theta}{2}\begin{cases}\tilde{l}\text{ for }j=\tilde{l}+½\\-\left(\tilde{l}+1\right)\text{ for }j=\tilde{l}-½\end{cases}\right), \end{aligned} \tag{70}$$

and

$$\begin{aligned} &E_{nc}^{ap}\left(n,q,m_0,m_1,\Theta,\tau,\chi,j,\tilde{l},\tilde{s},\widetilde{m}\right)\\ &=\frac{-q(b+q)-\sqrt{B_{nk}^{p2}-b(b+2q)}}{q^2+B_{nk}^{p2}}\\ &+\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,q,m_0,m_1)\\ &\times\left((\tau\aleph+\chi\Omega)m+\frac{\Theta}{2}\begin{cases}\tilde{l}\text{ for }j=\tilde{l}+½\\-\left(\tilde{l}+1\right)\text{ for }j=\tilde{l}-½\end{cases}\right). \end{aligned} \tag{71}$$

with

$$B_{nk}^{p}=n+½+\sqrt{(k-½)^2+b(b-2q)}$$

The first two parts in the right hand said of Eqs. (70) and (71) describe the ordinary relativistic energy of particle and anti-particle within the context of 3D-relativistic quantum mechanics known in the literature. The remaining terms describe the topological effect of the deformation space-space on these main energies in 3D-RQM regimes. While the corresponding physical expectation value $\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,q,m_0,m_1)$ can be obtained by applying the limit

$$\begin{aligned} &\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,q,m_0,m_1)\\ &=\underset{(C,H)\to(q,0)}{Lim}\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}\qquad(n,C,m_0,m_1,H) \end{aligned} \tag{72}$$

By assuming, in Eq. (58), the parameters $V_{sc}(r)=-S_{sc}(r)$ and $H=0$, i.e., $q_s=-q_v$ or $C_{ps}=0$ and $m_1=0$, one finds directly

$$\begin{aligned} &E_{nc}^{p}\left(n,q,m_0,\Theta,\tau,\chi,j,\tilde{l},\tilde{s},\widetilde{m}\right)\\ &=\frac{(n+k)^2-q^2}{(n+k)^2-q^2}m_0+\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,q,m_0)\\ &\times\left((\tau\aleph+\chi\Omega)\widetilde{m}+\frac{\Theta}{2}\begin{cases}\tilde{l}\text{ for }j=\tilde{l}+½\\-\left(\tilde{l}+1\right)\text{ for }j=\tilde{l}-½\end{cases}\right), \end{aligned} \tag{73}$$

and

$$\begin{aligned} &E_{nc}^{ap}\left(n,q,m_0,\Theta,\tau,\chi,j,\tilde{l},\tilde{s},\widetilde{m}\right)\\ &=-m_0+\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,q,m_0)\\ &\times\left((\tau\aleph+\chi\Omega)m+\frac{\Theta}{2}\begin{cases}\tilde{l}\text{ for }j=\tilde{l}+½\\-\left(\tilde{l}+1\right)\text{ for }j=\tilde{l}-½\end{cases}\right). \end{aligned} \tag{74}$$

The first two parts in the right hand said of Eqs. (73) and (74) describe the relativistic energy of particles and antiparticles within the context of RQM known. The remaining terms describe the topological effect of deformation space-space on these main energies in 3D-RQM regimes. While the expectation values $\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,q,m_0,m_1)$ can be obtained by applying the limit

$$\begin{aligned} &\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,q,m_0)\\ &=\underset{(C,H,m_1)\to(q,0,0)}{Lim}\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}(n,C,m_0,m_1,H) \end{aligned} \tag{75}$$

For the s-wave which corresponds $\tilde{l}=0$ or $k=1$ and $\widetilde{m}=0$, Eq. (72) reduced to

$$\begin{aligned} E_{nc}^{p}\left(n,q,m_0,\Theta,\tau,\chi,j,\tilde{l},\tilde{s},\widetilde{m}\right)&=-\frac{(n+1)^2-q^2}{(n+1)^2-q^2}m_0\\ &+\frac{\Theta}{2}\left\langle \widetilde{X}\right\rangle_{\left(n\tilde{l}\widetilde{m}\tilde{s}\right)}^{sc}\\ &\times(n,q,m_0)\begin{cases}0\text{ for }j=½\\-1\text{ for }j=\ -½\end{cases} \end{aligned} \tag{76}$$

The first part in the right hand said of Eq. (76) describes the relativistic energy of particles for the *s*-wave within the framework of 3D-RQM symmetries known in the literature. The remaining terms describe the topological effect of deformation space-space on these main energies in 3D-RQM symmetries.

4. IMPROVED SDM COULOMB POTENTIAL PROBLEM IN 3D-NRNCS SYMMETRIES

Two crucial steps must be taken to apply the improved spatially dependent mass Coulomb potential in 3-dimensional non-relativistic non-commutative space (3D-NRNCS) regimes to reach the non-relativistic limit. In standard non-relativistic quantum energy, the first step corresponds to the non-relativistic limit. Applying the steps that follow satisfies this as follows:

$$\begin{aligned} &(H,C_{sp})\to(0,0),\\ &E_{nk}^{s}+m_0\to 2\mu,\\ &E_{nk}^{s}-m_0\to E_{nl}^{nr},\\ &\quad k\to l\Rightarrow\gamma=m_0+E_{nk}^{s}-C_{sp}\to\gamma^{nr}=2\mu \end{aligned} \tag{77}$$

This permuted us to get the non-relativistic energy levels under SDM Coulomb potential in 3D-NRQM regimes as[4]

$$E_{nl}^{nr}=-\frac{\mu}{2}\frac{(q-b)^2}{\left(n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+b(b-2q)}\right)^2} \tag{78}$$

The second step corresponds to reporting the relativistic expectation values $\langle X\rangle_{(nlms)}^{sc}(n,C,m_0,m_1,H)$ of spin symmetry in Eq. (49) from the corresponding non-relativistic expectation values $\langle X\rangle_{(nlms)}^{sc}(n,q,m_0,m_1)$ as

$$\begin{aligned} &\langle X\rangle_{(nlms)}^{sc}(n,q,m_0,m)\\ &=(l(l+1)+m_1(m_1-q))\left\langle\frac{1}{r^4}\right\rangle_{(nlms)}^{s-sc}\\ &-(2\mu(m_1-q)-m_1(m_0-E_{nk}^{s}))\left\langle\frac{1}{r^3}\right\rangle_{(nlms)}^{s-sc} \end{aligned} \tag{79}$$

This allows us to obtain the non-relativistic correction energy $\Delta E_{nc-nr}^{sc}(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m)$ generated by the improved SDM Coulomb potential problems as

$$\begin{aligned} &\Delta E_{nc-nr}^{sc}(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m)\\ &=\langle X\rangle_{(nlms)}^{sc}(n,q,m_0,m)\\ &\times\left((\tau\aleph+\chi\omega)m+\frac{\Theta}{2}\begin{cases}l\text{ for }j=l+½\\-(l+1)\text{ for }j=\ l-½\end{cases}\right) \end{aligned} \tag{80}$$

The global nonrelativistic energy $E_{nc-nr}^{mt}(n,A,B,C,\alpha,\Theta,\tau,\chi,j,l,s,m)$ produced with the improved spatially dependent mass Coulomb potential in 3D-NRNCS regimes as a direct consequence of the deformation space-space is the sum of ordinary energy E_{nl}^{mt} in Eq. (78) under SDM Coulomb potential in 3D-NRQM symmetries plus the correction

$\Delta E^{mt}_{nc-nr}(n,q,m_0,m_1,\Theta,\tau,\chi,j,l,s,m)$ which we have seen in Eq. (80) as follows

$$\begin{aligned} & E^{sc}_{nc-nr} \\ & = -\frac{\mu}{2}\frac{(q-b)^2}{\left[n+½+\sqrt{(l+½)^2+b(b-2q)}\right]^2} \\ & +\langle X\rangle^{sc}_{(nlms)}(n,q,m_0,m) \\ & \times\left((\tau\aleph+\chi\omega)m+\frac{\Theta}{2}\begin{cases} l \text{ for } j=l+½ \\ -(l+1) \text{ for } j=\ l-½ \end{cases}\right) \end{aligned} \tag{81}$$

If we consider the case when $V_{sc}(r)=S_{sc}(r)$ and $H=0$, i.e., $q=q_v$ or $C_{sp}=0$ and $m_1=0$, then Eq. (81) can be simplified to the following expression

$$\begin{aligned} & E^{sc}_{nc-nr} \\ & = -\frac{\mu q^2}{2(n+l+1)^2} \\ & +\langle X\rangle^{sc}_{(nlms)}(n,q,m_0,m) \\ & \times\left((\tau\aleph+\chi\omega)m+\frac{\Theta}{2}\begin{cases} l \text{ for } j=l+½ \\ -(l+1) \text{ for } j=\ l-½ \end{cases}\right) \end{aligned} \tag{82}$$

The first part on the right-hand side of Eq. (82) traduces the non-relativistic energy of fermionic particles when mass becomes constant, which we note in Ref.[5] known in the literature while the remaining terms present the topological effect of deformation space-space on these principal energies. Through our observation of the newly obtained physical values within the framework of 3D-RNCQS and 3D-NRQS regimes, we observed that the obtained energy values $E^{sp}_{nc}(n,C,m_0,m_1,H,\Theta,\tau,\chi,j,l,s,m)$ and $E^{ps}_{nc}(n,C,m_0,m_1,H,\Theta,\tau,\chi,j,l,\tilde{l},\tilde{s},\widetilde{m})$ (see Eqs. (59) and (60)) are dependent on quantum numbers $\left(j,k,l,\tilde{l},s,\tilde{s},m,\widetilde{m}\right)$ in addition to the potential parameters (C,m_0,m_1,H). This means that the improved tensor interaction removes degeneracy between two states in the spin and pseudospin doublets. Since the total energy values have become dependent on the discrete $\left(j,k,l,\tilde{l},s,\tilde{s},m,\widetilde{m}\right)$, we can deduce that the new symmetry of 3D-RNCQS and 3D-NRQS has exact solutions in the first order of noncommutativity parameters (Θ,σ,χ), and also, the improved tensor interaction $U^{clp-s/p}_{eff}(r)$ removes degeneracy between two states in the spin and p-spin doublets. Furthermore, it is worth mentioning that the three-simultaneous limits $(\Theta,\sigma,\chi)\to(0,0,0)$, recover the equations of energy for the spin symmetry and the p-spin symmetry under an SDM Coulomb potential with Coulomb-like tensor interaction which is treated explicitly in main Refs.[4,5]

5. CONCLUSIONS

In the current work, the improved Coulomb-like tensor interaction under p-spin and spin symmetry limits with an arbitrary spin-orbit coupling quantum number k is used to approximate an analytical solution of the 3D-DDE with the newly spatially dependent mass Coulomb potential. We have obtained the new approximate bound state energies $E^{sp}_{nc}(n,C,m_0,m_1,H,\Theta,\tau,\chi,j,l,s,m)$ and $E^{ps}_{nc}(n,C,m_0,m_1,H,\Theta,\tau,\chi,j,l,\tilde{l},\tilde{s},\widetilde{m})$ for the H-atoms (He^{+}, Li^{+2}, Be^{3+}) that appeared to be sensitive to the quantum numbers $(j,k,l,\tilde{l},s,\tilde{s},m,\widetilde{m})$, the potential depths (n,C,m_0,m_1,H) of the studied potentials and non-commutativity parameters (Θ,σ,χ). We have additionally investigated a few different potentials that are pertinent to various physical systems. This treatment of the spatially dependent mass Coulomb potential's non-relativistic limit served as the culmination of our research as well. Additionally, a few special instances that readers and researchers found to be very concerning were dealt with within the context of 3D-RNCS symmetries. Furthermore, we achieved the non-relativistic limit of the improved spatially dependent mass Coulomb potential in 3D-NRNCS symmetries, which also includes the non-relativistic limit of the typical non-relativistic quantum energy and the influence of deformation space proportional to non-commutativity parameters and dimensionality of studied potential. In addition, we have shown that the non-commutativity of the coordinates that automatically appear, such as both perturbed spin-orbit interaction, pseudospin-orbit, new perturbed Zeeman Effect, and among others, and due to the behavior of topological properties of modified space-space, contributes to the physical effect of the four principals. These physical effects cannot be predicted using the relativistic quantum mechanics framework that has been described in the literature unless we take them into account as important concepts in the expression of the main Hamiltonian. These results have applications in chemical physics, atomic physics, and condensed matter physics. Notably, the standard physical values are recovered in all situations to produce the three simultaneous limits $(\Theta,\sigma,\chi)\to(0,0,0)$ in Refs.[4,5] which is treated within the framework of 3D-RQM and 3D-NRQM symmetries known in the literature.

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