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RESEARCH ARTICLE

A New Model for Non-relativistic Quantum Particles Interacting with IMMSEPM and IMSSKPM in 3D(NR)-NCPS Symmetries: Eigensolution and Thermodynamic Properties

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ABSTRACT

Using two physical models, we investigate the three-dimensional deformed Schrödinger equation (3D-DSE) in the context of three-dimensional non-relativistic noncommutative quantum phase-space (3D(NR)-NCPS) symmetries. The Mobius square potential is what they have in common. The first is the improved modified Mobius square plus Eckart potential (IMMSEPM), and the second is the improved Mobius square plus screened-Kratzer potential (IMSSKPM). This study solves the 3D-DSE in the 3D(NR)-NCPS regime using the well-known generalized Bopp's shifts method and conventional perturbation theory. For the homogeneous (K_2 and H_2) and heterogeneous (LiH, HCl, HF, DF, and CsO) diatomic molecules, the new non-relativistic energies equations and eigenfunction for the IMMSEPM and the IMSSKPM models in the presence of deformation phase-space are obtained to be sensitive to the atomic quantum numbers (j, l, s, m), the mixed potential depths (V_0, V_1, V_2, A, B) and (A, B, D_e, r_e), the screening parameter α, and non-commutativity parameters $(\Theta/\overline{\Phi}, \chi/\overline{\chi}, \zeta/\overline{\zeta})$ for the IMMSEPM and the IMSSKPM, respectively. By appropriately adjusting the improved modified Mobius square potential, the improved Eckart potential, the improved Hulthén potential, and the improved Poschl-Teller potential, we investigate the obtained new bound state eigenvalues of the DSE with the IMMSEPM and the IMSSKPM in 3D(NR)-NCPS symmetries. In addition, the partition function, mean energy, free energy, specific heat, and entropy of the IMMSEPM and IMSSKPM are investigated in 3D(NR)-NCPS symmetries. The current research has many potential applications in atomic and molecular physics.

Keywords: Schrödinger equation, Modified Mobius square plus Eckart potential, Mobius square plus screened-Kratzer potential, Noncommutative phase-space, Generalized Bopp's shift method

1. Introduction

Many indicators clearly show that the Schrödinger non-relativistic equation is still a significant method that allows the study of molecules subject to these potentials, such as modified Mobius squared potential (MSP), the Eckart potential (EP) and screened Kratzer potential (SKP), including all quantum information comprise of thermodynamic quantities. Onyenegecha et al. [1] obtained approximate relativistic and non-relativistic solutions for the modified MSP, which has the form $(-V_0(\frac{A+B\exp(-2\alpha r)}{1-\exp(-2\alpha r)})^2)$ using the NU method and investigating the behaviour of CO, NO, and HCl molecules subject to this potential. Durmuslar et al. [2] present a numerical study on modified MSP's electronic and optical properties in static electric and magnetic fields. Khordad and

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Ghanbari [3] investigated the thermal properties of the K_2 diatomic molecule using the generalized MSP and obtained this potential's rotational-vibrational energy levels analytically. Ikot et al. [4] solved the SE for the MSP using the NU functional analysis method. They investigated the Shannon entropy, Fisher information, Fisher-Shannon product, and the expectation values for this potential in position and momentum space for the low-lying states corresponding to $n = (0 \text{ and } 1)$. Okorie et al. [5] used the modified factorization approach to solve the SE with the modified MSP model and derive the energy eigenvalues equation and the related eigenfunction in terms of the hypergeometric function. Dong et al. [6] obtained the bound-state solutions of the SE with the Eckart potential $(-\frac{V_1 \exp(-2\alpha r)}{1-\exp(-2\alpha r)} + \frac{V_2 \exp(-2\alpha r)}{(1-\exp(-2\alpha r))^2})$ with the centrifugal term approximately. Falaye [7] used the asymptotic iteration method to determine any l-state solutions of the SE with the Eckart potential using proper centrifugal approximation. Ndjobo et al. [8] studied new solutions of the Bohr Hamiltonian model appropriate for triaxial nuclei with SKP, which has the form $(-2D_e(\frac{r_e}{r} - \frac{r_e^2}{2r^2})\exp(-\alpha r))$ in the framework of conformable fractional derivative and obtained the analytical solutions for energy spectra using the conformable fractional NU method. Ibekwe et al. [9] used the series expansion approach to solve the radial SE with the SKP. Inyang and Obisung [10] used the energy spectrum to generate eigenvalues for $X^3\sum^-$ state of *NI* and $X^1\sum^+$ state of ScI molecules with the SKP. Very recently (2023), Njoku et al. [11], the quantum information measures and complexity of the modified Mobius squared plus Eckart potential (MMSEP) were reported. Onyenegecha et al. [12] used the formula approach to get SE solutions for the MMPSE potential; they also reported numerical results and showed partition function and other thermodynamic features such as vibrational free energy, vibrational internal energy, vibrational entropy, and vibrational specific heat. Okon et al. 2021 [13] applied the NU method to calculate the bound state energies for two diatomic molecules such as Carbon (II) Oxide (CO) and Scandium Flouride (ScF) using Mobius square plus screen Kratzer potential (MSPSKP). They obtained the partition function and other thermodynamic properties of the diatomic molecules. Taş [14] (2023) determined the rovibrational energy values, Franck-Condon Factors, and r-centroids for Yttrium/Scandium oxides with ($B^2\Sigma^+$, $X^2\Sigma^+$) distinct electronic states using spectroscopic data. The bound state solution of the Schrödinger equation for the general molecular potential in D-dimensional space was used to calculate the rovibrational energy values. Very recently, Taş [15] (2024) obtained the bound state solutions of the Dirac equation by applying the Greene-Aldrich approach to the centrifugal term under spin/pseudo-spin symmetries for the ultra-generalized exponential hyperbolic potential model and gives numerical and graphical results bound states by using Mathematica Software package. Other works found in the literature concerned modified Mobius square plus Eckart potential [16, 17]. This paper aims to investigate the solutions of the 3-dimensional Schrodinger equation (SE) with the MMSEP model and MSPSKP model in the context of deformation phase-space, which were proposed in the 1930s by Heisenberg and Snyder in 1947 [18]. The main goal is to deepen previous studies on and investigate the possibility of discovering new applications within the framework of new extended symmetries, considered glasses that correct vision and make it more straightforward. To place both the specialized researcher and the reader, we refer to some previous work present in the literature related to the research topic within the framework of extended quantum mechanical symmetries, such as the improved MSP plus generalized Yukawa potential [19], modified EP plus a new modified deformed Hylleraas potential [20], the improved deformed generalized Deng-Fan potential plus deformed EP model [21], spatially-dependent mass Dirac equation (DE) for modified Hylleraas plus EP with improved Yukawa potential as a tensor interaction [22], the deformed DE in spatially dependent mass for the improved EP including the improved Yukawa tensor interaction [23], the deformed unequal scalar and vector Hellmann plus modified Kratzer potentials [24], arbitrary l-state with linear combination of Hulthén and Kratzer potentials [25], modified Kratzer potential [26] and the modified screened cosine Kratzer potential [27]. We will also benefit from them when we refer to them as special cases of our work later. The deformation phase-space or extended quantum mechanics known as non-commutative quantum mechanics (NCQM), which depends on additional hypotheses that we do not find in quantum mechanics known in the literature; the first hypothesis is related to the non-commuting of position-position: $(q_\mu^s, q_\mu^h, q_\mu^i) * (q_\nu^s, q_\nu^h, q_\nu^i)$, which is different from $(q_\nu^s, q_\nu^h, q_\nu^i) * (q_\mu^s, q_\mu^h, q_\mu^i)$ while the second $(\pi_\mu^s, \pi_\mu^h, \pi_\mu^i) * (\pi_\nu^s, \pi_\nu^h, \pi_\nu^i)$, which is different from $(\pi_\mu^s, \pi_\mu^h, \pi_\mu^i) * (\pi_\nu^s, \pi_\nu^h, \pi_\nu^i)$ is related to the non-commuting of momentum-momentum operators, in addition to the third usual hypothesis known in the literature $(q_\nu^s, q_\nu^h, q_\nu^i) * (\pi_\mu^s, \pi_\mu^h, \pi_\mu^i)$, which is different from $(\pi_\nu^s, \pi_\nu^h, \pi_\nu^i) * (q_\mu^s, q_\mu^h, q_\mu^i)$ that is, explain the non-commuting of (position-momentum) (For more details, see below) [28–36]. Research Connes [37–39]and Seiberg-Witten [40] has significantly impacted the developing concepts of NC quantum mechanics that were of scientific value and applicable to physical systems, especially in quantum field theory. My current work is motivated by these previous works. Based on the notable results

of Onyenegecha et al. and Ituen B. Okon et al. [12, 13], we are motivated to investigate the solutions to the deformed Schrödinger equations with the improved modified Mobius square plus Eckart potential (IMMSEPM) and improved Mobius square plus screened-Kratzer potential (IMSSKPM) in the context of three-dimensional non-relativistic noncommutative quantum phase-space (3D(NR)-NCPS) symmetries, in addition to showing the NC effect on the thermodynamic properties of the IMMSEPM and the IMSSKPM arising from the deformed phase-space. We do not know of any prior literature addressing this kind of study. We are the first to take on this significant subject in the context of non-relativistic quantum mechanics' extended symmetries. Our focus in this study will be on the ($V_{me}(q)$ and $V_{ms}(q)$) suggested IMMSEPM and IMSSKPM. Investigated collective potentials have been expressed in writing

$$V_{me}(q) = V_{me}(r) - \frac{\partial V_{me}(r)}{\partial r}\frac{\mathbf{L}.\mathbf{\Theta}}{2r} + \left(\Theta^2\right) \tag{1}$$

And a similar equation with the vector potential $V_{ms}(r)$. According to the view of 3D(NR)-QM symmetries that are known in the literature, the two vectors' potential ($V_{me}(r)$ and $V_{ms}(r)$) of the modified Mobius square plus Eckart potential (MMSEP) and Mobius square plus screened-Kratzer potential (MSSKP) are expressed as follows [12, 13]:

$$V_{me}(r) = -V_0\left(\frac{A + B\exp(-2\alpha r)}{1-\exp(-2\alpha r)}\right)^2 - \frac{V_1\exp(-2\alpha r)}{1-\exp(-2\alpha r)} + \frac{V_2\exp(-2\alpha r)}{(1-\exp(-2\alpha r))^2} \tag{2.1}$$

and

$$V_{ms}(r) = -V_0\left(\frac{A + B\exp(-\alpha r)}{1-\exp(-\alpha r)}\right)^2 - 2D_e\left(\frac{r_e}{r} - \frac{r_e^2}{2r^2}\right)\exp(-\alpha r)\,. \tag{2.2}$$

Here A is the potential range, B is molecular bond length, (V_0, V_1 and V_2) are parameters that describe the depth of potential well (D_e, $r_e, \alpha, q,$ and r) are the respective dissociation energy, equilibrium bond length, screening parameter, and the internuclear separations in the 3D(NC)PS and 3D(NR)-QM symmetries, respectively. The coupling $\mathbf{L}.\mathbf{\Theta}$ presents the scalar product of the usual components of the angular momentum operator $\mathbf{L}(L_x, L_y, L_z)$ and the infinitesimal non-commutativity vector $\mathbf{\Theta}(\theta_{12}, \theta_{23}, \theta_{13})/2$. For the NC quantum group, the noncentral generators can be suitably realized as self-adjoint differential operators that appear in three varieties ($q_\mu^{(s,h,i)}$ and $\pi_\nu^{(s,h,i)}$). In the representations of Schrödinger, Heisenberg, and interaction pictures, the first is the canonical structure variety (Csv), the second is the Lie structure variety (Lsv), and the final corresponds to the quantum plane variety (Qpv), satisfying the following deformed algebra [41–49]:

$$\left[x_\mu^s, p_\nu^s\right] = \left[x_\mu^h, p_\nu^h\right] = \left[x_\mu^i, p_\nu^i\right] = i\hbar\delta_{\mu\nu} \Rightarrow \left[q_\mu^s, \pi_\nu^s\right]_* = \left[q_\mu^h, \pi_\nu^{h,}\right]_* = \left[q_\mu^i, \pi_\nu^i\right]_* = i\hbar_{eff}\delta_{\mu\nu}, \tag{3}$$

$$\left[x_\mu^s, x_\nu^s\right] = \left[x_\mu^h, x_\nu^h\right] = \left[x_\mu^i, x_\nu^i\right] = 0 \Rightarrow \left[q_\mu^s, q_\nu^s\right]* = \left[q_\mu^h, q_\nu^h\right]_* = \left[q_\mu^i, q_\nu^i\right]_* = i\Lambda_{\mu\nu}, \tag{4}$$

$$\left[p_\mu^s, p_\nu^s\right] = \left[p_\mu^h, p_\nu^h\right] = \left[p_\mu^i, p_\nu^i\right] = 0 \Rightarrow \left[\pi_\mu{}^s, \pi_\nu^s\right]* = \left[\pi_\mu^h, \pi_\nu^h\right]_* = \left[\pi_\mu^i, \pi_\nu^i\right]_* = i\overline{\Phi}_{\mu\nu}. \tag{5}$$

where ($q_\mu^s, q_\mu^h, q_\mu^i$) and ($\pi_\mu^s, \pi_\mu^h, \pi_\mu^i$) are the deformed generalized coordinates and the corresponding deformed generalizing momentums, respectively, in 3D(NC)PS regimes, while the ($x_\mu^s, x_\mu^h, x_\mu^i$) and ($p_\mu^s, p_\mu^h, p_\mu^i$) are in the 3D(NR) QM symmetries, respectively, and

$\Lambda_{\mu\nu} = \varepsilon_{\mu\nu}\theta$ with $\varepsilon_{\mu\nu} \in$ Csv,

$\Lambda_{\mu\nu} = h_{\mu\nu}^\alpha q_\alpha^{(s,h,i)}$ with $h_{\mu\nu}^\alpha \in$ Lsv,

$\Lambda_{\mu\nu} = G_{\mu\nu}^{\alpha\beta} q_\alpha^{(s,h,i)} q_\beta^{(s,h,i)}$ with $G_{\mu\nu}^{\alpha\beta} \in$ Qpv.

Furthermore, when the uncertainty relation corresponding to the LHS of Eq. (3) is rewritten in 3D(NC)PS symmetries, it becomes:

$$\left|\Delta x_\mu^s \Delta p_\nu^s\right| = \left|\Delta x_\mu^h \Delta p_\nu^h\right| = \left|\Delta x_\mu^i \Delta p_\nu^i\right| \geq \hbar\delta_{\mu\nu}/2 \Rightarrow \left|\Delta q_\mu^s \Delta \pi_\nu^s\right| = \left|\Delta q_\mu^h \Delta \pi_\nu^h\right| = \left|\Delta q_\mu^i \Delta \pi_\nu^i\right| \geq \hbar_{eff}\delta_{\mu\nu}/2 \tag{6}$$

It is possible to express the multiple relationships observed in Eq. (6) with one connection, which we formulate as follows:

$$\left|\Delta x_\mu^{(s,h,i)} \Delta p_\nu^{(s,h,i)}\right| \geq \hbar\delta_{\mu\nu}/2 \Rightarrow \left|\Delta q_\mu^{(s,h,i)} \Delta \pi_\nu^{(s,h,i)}\right| \hbar_{eff}\alpha_{\mu\nu}/2 \tag{7}$$

However, the RHS of Eq. (4) reveals a new uncertainty relation:

$$\left(\left|\Delta q_\mu^{(s,h,i)} \Delta q_\nu^{(s,h,i)}\right|, \Delta \pi_\mu^{(s,h,i)} \Delta \pi_\nu^{(s,h,i)}\right) \geq \left(\frac{\theta\left|\varepsilon_{\mu\nu}\right|}{2}, \frac{\eta\left|\varepsilon_{\mu\nu}\right|}{2}\right) \text{ For Csv,}$$

$$\left(\left|\Delta q_\mu^{(s,h,i)} \Delta q_\nu^{(s,h,i)}\right|, \Delta \pi_\mu^{(s,h,i)} \Delta \pi_\nu^{(s,h,i)}\right) \geq \left(\frac{\beta_{\mu\nu}}{2} \frac{\gamma_{\mu\nu}}{2}\right), \text{ For Lsv,}$$

$$\left(\left|\Delta q_\mu^{(s,h,i)} \Delta q_\nu^{(s,h,i)}\right|, \left|\Delta \pi_\mu^{(s,h,i)} \Delta \pi_\nu^{(s,h,i)}\right|\right) \geq \left(\frac{L_{\mu\nu}}{2}, \frac{\delta_{\mu\nu}}{2}\right) \text{ For Qpv.} \tag{8}$$

with ($\beta_{\mu\nu}/\gamma_{\mu\nu}$ and $L_{\mu\nu}/\delta_{\mu\nu}$) are equal to the average values:

$$(\beta_{\mu\nu}, \gamma_{\mu\nu}) = \left(\left|\left\langle \sum_\alpha^3 f_{\mu\nu}^\alpha \pi_\alpha^{(s,h,i)} \right\rangle\right|, \left|\left\langle \sum_\alpha^3 f_{\mu\nu}^\alpha q_\alpha^{(s,h,i)} \right\rangle\right|\right)$$

and

$$(L_{\mu\nu}, \delta_{\mu\nu}) = \left(\left\langle \sum_{\alpha,\beta}^3 G_{\mu\nu}^{\alpha\beta} q_\alpha^{(s,h,i)} q_\beta^{(s,h,i)} \right\rangle, \left\langle \sum_{\alpha,\beta}^3 G_{\mu\nu}^{\alpha\beta} \pi_\alpha^{(s,h,i)} \pi_\beta^{(s,h,i)} \right\rangle\right)$$

It is essential to point out the new uncertainties we expressed in Eq. (7.2) and Eq. (8) have no similarity within the framework of non-relativistic quantum mechanics known in the literature (3D(NR)-QM symmetries). In 3D(NC)-NCPS symmetries, we prolonged the modified equal-time non-commutative canonical commutation relations to include Heisenberg, interaction pictures, and the usual Schrödinger picture. Here $\delta_{\mu\nu}$ is the Kronecker symbol, ($\theta_{\mu\nu}$ and $\overline{\Phi}_{\mu\nu}$) are antisymmetric real constants (3×3) matrices with $[\theta_{\mu\nu}]$ and $[\overline{\Phi}_{\mu\nu}]$ equal to (length square) and (momentums square), respectively, $\varepsilon_{\mu\nu}$ is an antisymmetric tensor operator describing the NC of space-time ($\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} = 1$ for $\mu \neq \nu$ and $\varepsilon_{\varepsilon\varepsilon} = 0$) and ($\theta, \eta$) are the noncommutative real parameters, while the effective Planck constant $\hbar_{eff}$ is comparable to the Planck constant $\hbar$. The new deformed scalar product between two arbitrary functions $f(x, p) * g(x, p)$ is defined by the Weyl-Moyal ($*$)-product for a canonical structure variety expressed as [48–53] (for simplicity, we have used the natural units $\hbar = c = 1$):

$$(f * g)(x, p) = (fg)(x, p) - \frac{i}{2}\theta^{\alpha\beta}\left(\partial_{x^\alpha} f \partial_{x^\beta} g\right)(x, p) - \frac{i}{2}\overline{\Phi}_{\mu\nu}\left(\partial_{p^\alpha} f \partial_{p^\beta} g\right)(x, p) + O\left(\overline{\Phi}^2, \theta^2\right) \tag{9}$$

here (∂_{x^α} and ∂_{p^α}) are equal to ($\frac{\partial}{\partial x^\alpha}$ and $\frac{\partial}{\partial p^\alpha}$), respectively. This allows the formation of two scales of space and phase cells with volumes ($\theta^{3/2}$ and $\overline{\Phi}^{3/2}$). The second component in Eq. (9) visually represents the consequences of space-space non-commutativity. In contrast, the third component in Eq. (9) physically represents the consequences of phase-phase non-commutativity. The remainder of the paper is outlined as follows: Section 2 presents an overview of the 3D-SE under the modified Mobius square plus Eckart potential model and Mobius square plus screened-Kratzer potential. Section 3 is devoted to investigating the 3D-DSE using the well-known Bopp's shifts method to obtain the effective potential for the IMMSEPM and the IMSSKPM. Furthermore, using standard perturbation theory, we find the expectation values of the radial terms ($\frac{s_1^{3/2}}{(1-s_1)^3}$, $\frac{s_1^{5/2}}{(1-s_1)^3}$, $\frac{s_1^{3/2}}{(1-s_1)^4}$, $\frac{s_1^{7/2}}{(1-s_1)^4}$, $\frac{s_1^{5/2}}{(1-s_1)^4}$, $\frac{s_1^{5/2}}{(1-s_1)^4}$, $\frac{s_1^{3/2}}{(1-s_1)^2}$ and $\frac{s_1^2}{(1-s_1)^4}$) and ($\frac{s_2}{(1-s_2)^3}$, $\frac{s_2^2}{(1-s_2)^3}$, $\frac{s_2}{(1-s_2)^4}$, $\frac{s_2^4}{(1-s_2)^4}$, $\frac{s_2^2}{(1-s_2)^4}$, $\frac{s_2}{(1-s_2)^4}$, $\frac{s_2}{(1-s_2)^2}$, $\frac{s_2}{(1-s_2)^3}$ and $\frac{1}{(1-s_2)^4}$) to

calculate the corrected relativistic and non-relativistic energy generated by the effect of the perturbed effective potentials $Z_{me}^{pert}(r)$ and $Z_{ms}^{pert}(r)$ of the IMMSEPM and IMSSKPM, and we derive the global corrected energies for bosonic particles/antiparticles, the homogeneous (K_2 and H_2) and heterogeneous LiH, HCl, HF, DF, and CsO diatomic molecules whose spin quantum number has an integer value. The following section will review the most important special cases, which will be valuable for specialists and readers. Section 5 uses the (IMMSEPM and IMSSKPM) to study homogeneous and heterogeneous composite systems in 3D(NR)-NCPS symmetries. Section 6 investigates the impact of phase-space deformation on the thermal parameters (partition function, mean energy, free energy, specific heat, and entropy) of the (IMMSEPM and IMSSKPM). A few succinct closing remarks are given in the last section.

2. A summary of SE in 3D(NR)-QM symmetry using the (MMSEP and MSSKP) in 3D(NR)-QM symmetry

Within the framework of the 3D(NR)-QM regime, it is helpful to remember the eigenvalues and corresponding eigenfunctions under the influence of the MMSEP and the MSSKP models to construct a physical model describing a physical system that interacted with the IMMSEPM and IMSSK. The two approaches are governed in this instance by the radial Schrödinger equations as follows:

$$\left(\frac{d^2}{dr^2}+2\mu\left(E_{nl}^{me}-V_{me}(r)-\frac{l(l+1)}{2\mu r^2}\right)\right)R_{nl}(r)=0 \tag{10}$$

And a similar equation with the vector potential $V_{ms}(r)$. The vectors $V_{me}(r)$ and $V_{ms}(r)$ are generated by the four-vector linear momentum operator $A^{\mu}(V_{me}(r)/V_{ms}(r), \mathbf{A}=0)$ and the reduced mass μ of (LiH, HCl, HF, DF, and CsO) molecule equal to $\frac{m_1 m_2}{m_1+m_2}$, m_1 and m_2 are the individual masses of the first and second atoms which constitute the molecules. The non-relativistic energy E_{nl}^{me} and E_{nl}^{ms} are dependent on the principal quantum number n and the orbital angular momentum quantum number l. The spherical symmetry of the MMSEP and MSSKP models permits the wave function solutions $\Psi^{(1)}(r,\Omega_3)$ and $\Psi^{(2)}(r,\Omega_3)$ of the recognized forms $\frac{R_{nl}/U_{nl}(r)}{r}Y_m^l(\Omega_3)$ while $Y_m^l(\Omega_3)$, where m represents the projections on the (Oz) axis and $Y_m^l(\Omega_3)$ are spherical harmonics. In usual 3D-RQM symmetries, the Refs. Authors [12, 13] use the formula method and the NU method, respectively, to derive the expression of the wave functions $\Psi^{(1)}(r,\Omega_3)$ and $\Psi^{(2)}(r,\Omega_3)$ as functions of the generalized hypergeometric functions ${}_2F_1(-n, 2\sqrt{-\Lambda_{nl}}+1+\chi_l; 2\sqrt{-\Lambda_{nl}}+1, s_1)$ and ${}_2F_1(-n, n+2\Omega_{1nl}+2\Omega_{2l}+1; 1+2\Omega_{1nl}, s_2)$ as:

$$\Psi^{(1)}(r,\Omega_3)=N_n\frac{s_1^{\sqrt{-\Lambda_{nl}}}(1-s_1)^{\frac{1}{2}(1+\chi_l)}}{r}Y_l^m(\Omega_3)$$
$${}_2F_1\left(-n, 2\sqrt{-\Lambda_{nl}}+1+\chi_l; 2\sqrt{-\Lambda_{nl}}+1, s_1\right) \tag{11}$$

and

$$\Psi^{(2)}(r,\Omega_3)=A_{nl}\frac{s_2^{\Omega_{1nl}}(1-s_2)^{\left(\frac{1}{2}+\Omega_{2l}\right)}}{r}Y_l^m(\Omega_3)$$
$${}_2F_1\left(-n, n+2\Omega_{1nl}+2\Omega_{2l}+1; 1+2\Omega_{1nl}, s_2\right) \tag{12}$$

With

$$s_1=\exp(-2\alpha r), s_2=\exp(-\alpha r),$$

$$\Lambda_{nl}=\frac{\mu}{2\alpha^2}\left(E_{nl}^{me}+V_0A^2\right),$$

$$\chi_l=\sqrt{(2l+1)^2+\frac{2\mu}{\alpha^2}(V_2+V_0(A+B))},$$

$$\Omega_{1nl}=\sqrt{-\frac{2E_{nl}^{ms}}{\alpha^2}+\frac{2\mu V_0A^2}{\alpha^2}+l\left(l+1\right)},$$

$$\Omega_{2l}=\sqrt{\frac{2\mu V_0(A+B)^2}{\alpha^2}+2\mu D_e r_e^2+\left(l+\frac{1}{2}\right)^2},$$

and

$$A_{nl}=N_{nl}\frac{\Gamma\left(n+2\Omega_1+1\right)}{n!\Gamma\left(2\Omega_1+1\right)}$$

We have used the definition of Jacobi polynomials [54]:

$$P_n^{(A_n,B_n)}\left(1-2x\right)=\frac{\Gamma\left(n+A_n+1\right)}{n!\Gamma\left(A_n+1\right)}\,{}_2F_1\left(-n,n+A_n+B_n+1;1+A_n,x\right) \tag{13}$$

While N_n and N_{nl} are normalization constants. The corresponding non-relativistic energies E_{nl}^{me} and E_{nl}^{ms} of the MMSEP and the MSSKP models are given by

$$E_{nl}^{me}=a-q\left(\frac{b}{n+\sigma}+\frac{n+\sigma}{2}\right)^2, \tag{14}$$

and

$$E_{nl}^{ms}=V_0A^2+\frac{\alpha^2l\left(l+1\right)}{2\mu}-\frac{\alpha^2}{8\mu}\left(n+\Delta+\frac{Q_2}{n+\Delta}\right)^2\equiv-\left(Q_1(n+\Delta)^2+\frac{Q_1Q_2^2}{(n+\Delta)^2}\right)-\Lambda_0. \tag{15}$$

with $a=-V_0A^2$, $q=\frac{2\alpha^2}{\mu}$, $b=\frac{\mu(V_1-V_2+2V_0(A+B))}{2\alpha^2}+\sigma(1-\sigma)$, $\sigma=\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{2\mu(V_2-V_0(A+B)^2)}{\alpha^2}}$, $\Lambda_0=2Q_1Q_2-Q_0$, $Q_0=V_0A^2+\frac{\alpha^2l(l+1)}{2\mu}$, $Q_1=\frac{\alpha^2}{8\mu}$, $Q_2=\frac{2\mu V_0(A^2-B^2)}{\alpha^2}-\frac{4\mu D_e r_e}{\alpha}+l(l+1)$ and $=\frac{1}{2}+\sqrt{\frac{2\mu V_0(A+B)^2}{\alpha^2}+2\mu D_e r_e^2+(l+1/2)^2}$.

We will examine the new deformed non-relativistic quantum theory in 3D(NR)-NCPS symmetries and introduce a new deformed combination potential, taking into account the IMMSEPM and IMSSKPM in the following section.

3. Solutions of DSE under the IMMSEPM and the IMSSKPM in 3D(NR)-NCPS

3.1. Apply GBSM to the IMMSEPM and the IMSSKPM

In this subsection, we will discuss the essential methods for resolving the effect of non-relativistic noncommutative phase-space on the SE in 3D(NR)-NCPS symmetries using the *IMMSEPM* and the *IMSSKPM*. Our goal is accomplished by applying the new concepts introduced in the introduction, Eqs. (3.1), (4.1), (5.1), and Eq. (9), which are expressed in new relationships defined by modified noncommutative canonical commutation relations and the concept of the Weyl-Moyal star product. With these data, we can rewrite the standard radial SE in Eq. (10) in 3D(NR)-NCPS as follows:

$$\left(\frac{d^2}{dr^2}+2\mu\left(E_{nl}^{me}+V_0\left(\frac{A+B\exp\left(-2\alpha r\right)}{1-\exp\left(-2\alpha r\right)}\right)^2+\frac{V_1\exp\left(-2\alpha r\right)}{1-\exp\left(-2\alpha r\right)}-\frac{V_2\exp\left(-2\alpha r\right)}{\left(1-\exp\left(-2\alpha r\right)\right)^2}-\frac{l\left(l+1\right)}{2\mu r^2}\right)\right)*R_{nl}\left(r\right)=0 \tag{16}$$

And a similar equation with the vector potential $V_{ms}(r)$. Quantum field theory may include non-commutativity in two ways: In the first method, which is similar to the Taylor development [55–60], the various NC physical fields ($\boldsymbol{\Psi_{nl}}$, $\boldsymbol{\Gamma_{nl}}$, $\boldsymbol{e_\mu^a}$, $\boldsymbol{F_{\alpha\beta}}$, ...) are rewritten in terms of their corresponding fields (Ψ_{nl}, Γ_{nl}, e_μ^a, $F_{\alpha\beta}$, ...) in the known quantum space in the literature, in proportion to the NC-(phase-space) parameters $\boldsymbol{\Theta}(\theta_{12},\theta_{23},\theta_{13})/2$

and $\overline{\Phi}(\overline{\Phi}_{12}, \overline{\Phi}_{23}, \overline{\Phi}_{13})$. The second approach, however, requires reformulating the non-commutative operator (q, π) considering the features of the quantum operators (x, p) that are known from the literature and the spatial characteristics connected to the NC-(phase-space) parameters $\Theta(\theta_{12}, \theta_{23}, \theta_{13})/2$ and $\overline{\Phi}(\overline{\Phi}_{12}, \overline{\Phi}_{23}, \overline{\Phi}_{13})$. Various approaches to solving a physics problem speed up the process rather than altering the result. Concerning our existing situations, both approaches produce the same results. Specialist researchers know that F. Bopp introduced the generalized Bopp's shifts method (GBSM) [61–65], which is a new quantization rule $(x, p) \rightarrow (q = x - \frac{i}{2}\partial_p, \pi = p + \frac{i}{2}\partial_x)$ as an alternative to the standard correspondence $(x, p) \rightarrow (q = x, \pi = p + \frac{i}{2}\partial_x)$. This quantization procedure is called Bopp quantization [64]. The Weyl-Moyal star product $G(x, p) * K(x, p)$ induces GBSM in the respect that it is replaced by $G\left(x - \frac{i}{2}\partial_p, p + \frac{i}{2}\partial_x\right) * K(x, p)$ [65]. It helps us to achieve

$$\left(V_{me}(r) * R_{nl}(r),\ V_{ms}(r) * U_{nl}(r)\right) = \left(V_{me}(q) R_{nl}(r), V_{ms}(q) U_{nl}(r)\right)$$

$$\frac{l(l+1)}{2\mu r^2} * \left(R_{nl}(r)/U_{nl}(r)\right) = \frac{l(l+1)}{2\mu q^2}\left(R_{nl}(r)/U_{nl}(r)\right)$$

$$\frac{p^2}{2\mu} * \left(R_{nl}(r)/U_{nl}(r)\right) = \frac{\pi^2}{2\mu}\left(R_{nl}(r)/U_{nl}(r)\right),$$

With

$$V_{me}(q) = -V_0\left(\frac{A + B\exp(-2\alpha q)}{1 - \exp(-2\alpha q)}\right)^2 - \frac{V_1 \exp(-2\alpha q)}{1 - \exp(-2\alpha q)} + \frac{V_2 \exp(-2\alpha q)}{(1 - \exp(-2\alpha q))^2} \tag{17}$$

And a similary equation with the vector potential $V_{ms}(q)$. When applied by specialists to the four fundamental equations that correspond to the non-relativistic SE [66–73], and the other three relativistic equations represented by the relativistic KGE [72, 74–80], the relativistic Dirac equation [81–87], and the Duffin-Kemmer-Petiau equation [88–91], the GBSM has demonstrated remarkable success. It is important to note that GBSM enables us to reduce Eqs. (16.1) and (16.2) to their new form:

$$\left(\frac{d^2}{dr^2} + 2\mu\left(E_{nl}^{me} + V_0\left(\frac{A + B\exp(-2\alpha q)}{1 - \exp(-2\alpha q)}\right)^2 + \frac{V_1 \exp(-2\alpha q)}{1 - \exp(-2\alpha q)} - \frac{V_2 \exp(-2\alpha q)}{(1 - \exp(-2\alpha q))^2} - \frac{l(l+1)}{2\mu q^2}\right)\right) R_{nl}(r) = 0 \tag{18}$$

And a similary equation with the vector potential $V_{ms}(q)$. The modified algebraic structure of non-commutative covariant canonical commutation relations with the notion of the Weyl-Moyal star product in Eq. (4) and Eq. (8) yield the corresponding algebra known products in the literature that correspond to SP, HP, and IP, respectively [61–65]:

$$\left(\left[q_\mu^{(s,h,i)}, \pi_\nu^{(s,h,i)}\right], \left[\pi_\mu^{(s,h,i)}, \pi_\nu^{(s,h,i)}\right]\right) = \left(i\hbar_{eff}\delta_{\mu\nu}, i\theta_{\mu\nu}, i\overline{\Phi}_{\mu\nu}\right) \tag{19}$$

The deformed generalized coordinates (q_μ^s, q_μ^h and q_μ^i) and the corresponding deformed generalizing momentums (π_μ^s, π_μ^h and π_μ^i), in 3D(NC)-PS symmetries, are expressed as a function on corresponding values (x_μ^s, x_μ^h and x_μ^i) and (p_μ^s, p_μ^h and p_μ^i) in 3D-NRQSM symmetries, as follows:

$$\left(q_\mu^s, \pi_\mu^s\right) = \left(x_\mu^s - \sum_{\nu=1}^{3}\frac{\theta_{\mu\nu}}{2}p_\nu^s + \left(\Theta^2\right),\ p_\mu^s + \sum_{\beta=1}^{3}\frac{\overline{\Phi}_{\mu\beta}}{2}x_\beta^s + O\left(\overline{\Phi}^2\right)\right) \tag{20.1}$$

and

$$\left(q_\mu^h(t), \pi_\mu^h(t)\right) = \left(x_\mu^h(t) - \sum_{\nu=1}^{3}\frac{\theta_{\mu\nu}}{2}p_\nu^h(t) + \left(\Theta^2\right), p_\mu^h(t) + \sum_{\beta=1}^{3}\frac{\overline{\Phi}_{\mu\beta}}{2}x_\beta^h(t) + O\left(\overline{\Phi}^2\right)\right) \tag{20.2}$$

and

$$\left(q_{\mu}^{i}(t),\pi_{\mu}^{i}(t)\right)=\left(x_{\mu}^{i}(t)-\sum_{\nu=1}^{3}\frac{\theta_{\mu\nu}}{2}p_{\nu}^{i}(t)+O\left(\Theta^{2}\right),p_{\mu}^{i}(t)+\sum_{\beta=1}^{3}\frac{\overline{\Phi}_{\mu\beta}}{2}x_{\beta}^{i}(t)+O\left(\overline{\Phi}^{2}\right)\right) \tag{20.3}$$

These data permuted us to obtain the modified operators (q^2, $\frac{l(l+1)}{2\mu q^2}$, π^2 and ($V_{me}(q)$, $V_{ms}(q)$), in the 3D(NR)-NCPS regimes:

$$q^{2}=r^{2}-\mathbf{L}.\boldsymbol{\Theta}+O\left(\Theta^{2}\right),$$

$$\frac{l\left(l+1\right)}{2\mu q^{2}}=\frac{l\left(l+1\right)}{2\mu r^{2}}+\frac{l\left(l+1\right)}{2\mu r^{4}}\mathbf{L}.\boldsymbol{\Theta}+O\left(\Theta^{2}\right),$$

$$V_{me}(q)=-V_{0}\left(\frac{A+B\exp\left(-2\alpha r\right)}{1-\exp\left(-2\alpha r\right)}\right)-\frac{V_{1}\exp\left(-2\alpha r\right)}{1-\exp\left(-2\alpha r\right)}+\frac{V_{2}\exp\left(-2\alpha r\right)}{\left(1-\exp\left(-2\alpha r\right)\right)^{2}}-\frac{1}{2r}\frac{\partial V_{me}}{\partial r}\mathbf{L}.\boldsymbol{\Theta}+O\left(\Theta^{2}\right)$$

$$\pi^{2}=p^{2}+\mathbf{L}.\overline{\boldsymbol{\Phi}}+O\left(\overline{\Phi}^{2}\right). \tag{21}$$

And a similary equation with the vector potential $V_{ms}(q)$. The new coupling $\mathbf{L}.\ \overline{\Phi}$ is $L_x\eta_{12}+L_y\eta_{23}+L_z\eta_{13}$. By translating Eq. (21) into Eqs. (18.1) and (18.2), we may get the like-SE that follows:

$$\left(\frac{d^{2}}{dr^{2}}+2\mu\left(E_{nl}^{me}+V_{0}\left(\frac{A+B\exp\left(-2\alpha r\right)}{1-\exp\left(-2\alpha r\right)}\right)+\frac{V_{1}\exp\left(-2\alpha r\right)}{1-\exp\left(-2\alpha r\right)}-\frac{V_{2}\exp\left(-2\alpha r\right)}{\left(1-\exp\left(-2\alpha r\right)\right)^{2}}-\frac{l\left(l+1\right)}{2\mu r^{2}}\right)+2\mu Z_{me}^{pert}(r)\right)$$
$$\times\ R_{nl}(r)=0 \tag{22}$$

And a similar DSE equation with the vector potential $V_{ms}(r)$. With

$$Z_{me}^{pert}(r)=\left(\frac{1}{2r}\frac{\partial V_{me}}{\partial r}-\frac{l\left(l+1\right)}{2\mu r^{4}}\right)\mathbf{L}.\boldsymbol{\Theta}+O\left(\Theta^{2}\right) \tag{23}$$

And a similar equation with the vector potential $Z_{ms}^{pert}(r)$. These equations combine the physical properties of the two potentials under study, the *IMMSEPM* and the *IMSSKPM*, as well as the topological properties of the naturally deformed space. Performing the calculations, one gets ($\frac{1}{2r}\frac{\partial V_{me}(r)}{\partial r}$, $\frac{1}{2r}\frac{\partial V_{ms}(r)}{\partial r}$) and $\frac{\pi^2}{2\mu}$ as follows:

$$\begin{aligned}\frac{1}{2r}\frac{\partial V_{me}(r)}{\partial r}&=\frac{T_{1}\exp\left(-2\alpha r\right)}{r(1-\exp\left(-2\alpha r\right))^{2}}+\frac{T_{2}\exp\left(-4\alpha r\right)}{r(1-\exp\left(-2\alpha r\right))^{2}}\\&+\frac{T_{3}\exp\left(-2\alpha r\right)}{r(1-\exp\left(-2\alpha r\right))^{3}}+\frac{T_{4}\exp\left(-6\alpha r\right)}{r(1-\exp\left(-2\alpha r\right))^{3}}+\frac{T_{5}\exp\left(-4\alpha r\right)}{r(1-\exp\left(-2\alpha r\right))^{3}}\end{aligned} \tag{24}$$

and

$$\begin{aligned}\frac{1}{2r}\frac{\partial V_{ms}(r)}{\partial r}&=\frac{L_{1}\exp\left(-\alpha r\right)}{r(1-\exp\left(-\alpha r\right))^{2}}+\frac{L_{2}\exp\left(-2\alpha r\right)}{r(1-\exp\left(-\alpha r\right))^{2}}+\frac{L_{3}\exp\left(-\alpha r\right)}{r(1-\exp\left(-\alpha r\right))^{3}}+\frac{L_{4}\exp\left(-3\alpha r\right)}{r(1-\exp\left(-\alpha r\right))^{3}}\\&+\frac{L_{5}\exp\left(-2\alpha r\right)}{r(1-\exp\left(-\alpha r\right))^{3}}+L_{6}\frac{\exp\left(-\alpha r\right)}{2r^{4}}+L_{7}\frac{\exp\left(-\alpha r\right)}{r^{2}}+L_{8}\frac{\exp\left(-\alpha r\right)}{2r^{3}},\end{aligned} \tag{25}$$

With

$$T_{1}=\left(2V_{0}BA\alpha-V_{2}\right),\ T_{2}=\left(2V_{0}B^{2}+V_{1}\right)\alpha,\ T_{3}=2\alpha V_{0}A^{2},\ T_{4}=2\alpha V_{0}B^{2},\ T_{5}=2\left(2V_{0}AB-V_{2}\right)\alpha,\ T_{6}=\alpha V_{1},$$

$$L_{1}=-\alpha V_{0}AB,\ L_{2}=-\alpha V_{0}B^{2},\ L_{3}=-\alpha V_{0}A^{2},\ L_{4}=-\alpha V_{0}B^{2},\ L_{5}=-2\alpha V_{0}AB,\ L_{6}=-2D_{e}r_{e}^{2}\text{ and }L_{7}=\alpha D_{e}r_{e}$$

and $L_{8}=D_{e}\left(2-\alpha r_{e}\right)r_{e}$

While

$T_1 = (2V_0BA\alpha - V_2)$, $T_2 = \left(2V_0B^2 + V_1\right)\alpha$, $T_3 = 2\alpha V_0A^2$, $T_4 = 2\alpha V_0B^2$, $T_5 = 2\left(2V_0AB - V_2\right)\alpha$, $T_6 = \alpha V_1$, $L_1 = -\alpha V_0AB$, $L_2 = -\alpha V_0B^2$, $L_3 = -\alpha V_0A^2$, $L_4 = -\alpha V_0B^2$, $L_5 = -2\alpha V_0AB$, $L_6 = -2D_er_e^2$ and $L_7 = \alpha D_er_e$ and $L_8 = D_e\left(2 - \alpha r_e\right)r_e$

Substituting Eq. (24) into Eq. (23), the spontaneously generated terms ($V^{me}_{eff-nc}(r,\Theta) \equiv V^{me}_{eff-nc}$ and $V^{se}_{eff-nc}(r,\Theta) \equiv V^{se}_{eff-nc}$) and the global working Hamiltonian operators $H^{me}_{nc}(r,\pi,\Theta,\overline{\Phi}) \equiv H^{me}_{nc}$ and $H^{ms}_{nc}(r,\pi,\Theta,\overline{\Phi}) \equiv H^{ms}_{nc}$) of the IMMSEPM and IMSSKPM, as a result of the topological properties of the deformation space-space, can be expressed as

$$\left(V^{me}_{eff-nc}, V^{ms}_{eff-nc}\right) = \left(V^{me}_{eff}(r) + Z^{pert}_{me}, V^{ms}_{eff}(r) + Z^{pert}_{ms}\right)$$

and

$$\left(H^{me}_{nc}, H^{ms}_{nc}\right) = H_{me}(p,x) + H^{me}_{pert}, H_{ms}(p,x) + H^{ms}_{pert}) \tag{26}$$

with

$$H^{me}_{pert}\left(r,\pi,\Theta,\overline{\Phi}\right) = \frac{\mathbf{L}.\overline{\mathbf{\Phi}}}{2\mu} + \left(\frac{T_1\exp(-2\alpha r)}{r(1-\exp(-2\alpha r))^2} + \frac{T_2\exp(-4\alpha r)}{r(1-\exp(-2\alpha r))^2} + \frac{T_3\exp(-2\alpha r)}{r(1-\exp(-2\alpha r))^3} + \frac{T_4\exp(-6\alpha r)}{r(1-\exp(-2\alpha r))^3} + \frac{T_5\exp(-4\alpha r)}{r(1-\exp(-2\alpha r))^3} + \frac{T_6\exp(-2\alpha r)}{r\left(1-\exp(-2\alpha r)\right)} - \frac{l(l+1)}{2\mu r^4}\right)\mathbf{L}\mathbf{\Theta} + O\left(\Theta^2,\overline{\Phi}^2\right) \tag{27}$$

and

$$H^{ms}_{pert}\left(r,\pi,\Theta,\overline{\Phi}\right) = \frac{\mathbf{L}.\overline{\mathbf{\Phi}}}{2\mu} + \left(\frac{L_1\exp(-\alpha r)}{r(1-\exp(-\alpha r))^2} + \frac{L_2\exp(-2\alpha r)}{r(1-\exp(-\alpha r))^2} + \frac{L_3\exp(-\alpha r)}{r(1-\exp(-\alpha r))^3} + \frac{L_4\exp(-3\alpha r)}{r(1-\exp(-\alpha r))^3} + \frac{L_5\exp(-2\alpha r)}{r(1-\exp(-\alpha r))^3} + \frac{L_6\exp(-\alpha r)}{2r^4} + \frac{L_7\exp(-\alpha r)}{r^2} + \frac{L_8\exp(-\alpha r)}{2r^3} - \frac{l(l+1)}{2\mu r^4}\right)\mathbf{L}.\mathbf{\Theta} + O\left(\Theta^2,\overline{\Phi}^2\right) \tag{28}$$

and

$$Z^{pert}_{me}(r) = \left(\frac{T_1\exp(-2\alpha r)}{r(1-\exp(-2\alpha r))^2} + \frac{T_2\exp(-4\alpha r)}{r(1-\exp(-2\alpha r))^2} + \frac{T_3\exp(-2\alpha r)}{r(1-\exp(-2\alpha r))^3} + \frac{T_4\exp(-6\alpha r)}{r(1-\exp(-2\alpha r))^3} + \frac{T_5\exp(-4\alpha r)}{r(1-\exp(-2\alpha r))^3} + \frac{T_6\exp(-2\alpha r)}{r\left(1-\exp(-2\alpha r)\right)} - \frac{l(l+1)}{2\mu r^4}\right)\mathbf{L}.\mathbf{\Theta} + O\left(\Theta^2\right) \tag{29}$$

and

$$Z^{pert}_{ms}(r) = \left(\frac{L_1\exp(-\alpha r)}{r(1-\exp(-\alpha r))^2} + \frac{L_2\exp(-2\alpha r)}{r(1-\exp(-\alpha r))^2} + \frac{L_3\exp(-\alpha r)}{r(1-\exp(-\alpha r))^3} + \frac{L_4\exp(-3\alpha r)}{r(1-\exp(-\alpha r))^3} + \frac{L_5\exp(-2\alpha r)}{r(1-\exp(-\alpha r))^3} + \frac{L_6\exp(-\alpha r)}{2r^4} + \frac{L_7\exp(-\alpha r)}{r^2} + \frac{L_8\exp(-\alpha r)}{2r^3} - \frac{l(l+1)}{2\mu r^4}\right)\mathbf{L}.\mathbf{\Theta} + O\left(\Theta^2\right) \tag{30}$$

$$H_{me}(p,x) = \frac{p^2}{2\mu} - V_0\left(\frac{A + B\exp(-2\alpha r)}{1-\exp(-2\alpha r)}\right) - \frac{V_1\exp(-2\alpha r)}{1-\exp(-2\alpha r)} + \frac{V_2\exp(-2\alpha r)}{(1-\exp(-2\alpha r))^2} \tag{31}$$

and

$$H_{ms}(p,x)=\frac{p^2}{2\mu}+V_0\left(\frac{A+B\exp(-\alpha r)}{1-\exp(-\alpha r)}\right)-2D_e\left(\frac{r_e}{r}-\frac{r_e^2}{2r^2}\right)\exp(-\alpha r) \tag{32}$$

We can express the global effective potential in 3D(NR)-NCPS symmetries $V_{me}^{nc-eff}(r,\Theta)$ and $V_{ms}^{nc-eff}(r,\Theta)$ as a function of corresponding follows of the effective potential $V_{me}^{eff}(r)$ and $V_{ms}^{eff}(r)$, respectively, in 3D(NR)-QM symmetries as:

$$\begin{aligned}V_{me}^{nc-eff}(r,\Theta)=&-V_0\left(\frac{A+B\exp(-2\alpha r)}{1-\exp(-2\alpha r)}\right)-\frac{V_1\exp(-2\alpha r)}{1-\exp(-2\alpha r)}+\frac{V_2\exp(-2\alpha r)}{(1-\exp(-2\alpha r))^2}+\frac{l(l+1)}{2\mu r^2}\\&-\left(\frac{l(l+1)}{2\mu r^4}-\frac{1}{2r}\frac{\partial V_{me}(r)}{\partial r}\right)\mathbf{L}.\boldsymbol{\Theta}+O(\Theta^2)\end{aligned} \tag{33}$$

and

$$\begin{aligned}V_{ms}^{nc-eff}(r,\Theta)=&V_0\left(\frac{A+B\exp(-\alpha r)}{1-\exp(-\alpha r)}\right)-2D_e\left(\frac{r_e}{r}-\frac{r_e^2}{2r^2}\right)\exp(-\alpha r)+\frac{l(l+1)}{2\mu r^2}-\left(\frac{l(l+1)}{2\mu r^4}-\frac{1}{2r}\frac{\partial V_{ms}(r)}{\partial r}\right)\mathbf{L}.\boldsymbol{\Theta}\\&+O(\Theta^2)\end{aligned} \tag{34}$$

Furthermore, Eq. (22) cannot be solved analytically for any state $l\neq 0$ because of the centrifugal terms ($\frac{l(l+1)}{r^2}$, $\frac{l(l+1)}{r^4},\ldots$) and the studied potential itself. The effective potentials $V_{me}^{nc-eff}(r,\Theta)$ and $V_{ms}^{nc-eff}(r,\Theta)$ given in Eqs. (33) and (34) have a strong singularity $r\to 0$, we need to use the suitable improved approximation of the centrifugal term proposed by Greene and Aldrich [92] and applied by Onyenegecha et al. and Ituen B. Okon et al. [12, 13]. The radial part of the 3D-DSE with IMMSEPM and IMSSKPM contains the centrifugal terms $\frac{l(l+1)}{r^2}$ and $\frac{l(l+1)}{r^4}$ since we assume $l\neq 0$. However, the IMMSEPM and the IMSSKPM are a kind of potentials that cannot be solved precisely when the centrifugal term is taken into account unless $l=0$ is assumed. The conventional approximations used in this work for the IMMSEPM and the IMSSKPM, respectively:

$$\begin{aligned}\frac{1}{r^2}&\approx\frac{4\alpha^2\exp(-2\alpha r)}{(1-\exp(-2\alpha r))^2}=\frac{4\alpha^2 s_1}{(1-s_1)^2},\\\frac{1}{r^2}&\approx\frac{\alpha^2}{(1-\exp(-\alpha r))^2}=\frac{\alpha^2}{(1-s_2)^2}.\end{aligned} \tag{35}$$

After appropriate calculations, we can find the following results:

$$\begin{aligned}&\frac{\exp(-2\alpha r)}{r(1-\exp(-2\alpha r))^2}=\frac{2\alpha s_1^{\frac{3}{2}}}{(1-s_1)^3},\quad\frac{\exp(-4\alpha r)}{r(1-\exp(-2\alpha r))^2}=\frac{2\alpha s_1^{5/2}}{(1-s_1)^3},\quad\frac{\exp(-2\alpha r)}{r(1-\exp(-2\alpha r))^3}=\frac{2\alpha s_1^{\frac{3}{2}}}{(1-s_1)^4},\\&\frac{\exp(-6\alpha r)}{r(1-\exp(-2\alpha r))^3}=\frac{2\alpha s_1^{7/2}}{(1-s_1)^4},\quad\frac{\exp(-4\alpha r)}{r(1-\exp(-2\alpha r))^3}=\frac{2\alpha s_1^{\frac{5}{2}}}{(1-s_1)^4},\quad\frac{\exp(-2\alpha r)}{r(1-\exp(-2\alpha r))}=\frac{2\alpha s_1^{3/2}}{(1-s_1)^2},\\&\frac{l(l+1)}{2\mu r^4}=\frac{16l(l+1)\alpha^4 s_1^2}{2\mu(1-s_1)^4}\end{aligned} \tag{36}$$

and

$$\begin{aligned}&\frac{\exp(-\alpha r)}{r(1-\exp(-\alpha r))^2}=\alpha\frac{s_2}{(1-s_2)^3},\quad\frac{\exp(-2\alpha r)}{r(1-\exp(-\alpha r))^2}=\alpha\frac{s_2^2}{(1-s_2)^3},\quad\frac{\exp(-\alpha r)}{r(1-\exp(-\alpha r))^3}=\alpha\frac{s_2}{(1-s_2)^4},\\&\frac{\exp(-3\alpha r)}{r(1-\exp(-\alpha r))^3}=\alpha\frac{s_2^4}{(1-s_2)^4},\quad\frac{\exp(-2\alpha r)}{r(1-\exp(-\alpha r))^3}=\alpha\frac{s_2^2}{(1-s_2)^4},\quad\frac{\exp(-\alpha r)}{2r^4}=\frac{\alpha^4}{2}\frac{s_2}{(1-s_2)^4},\end{aligned}$$

$$\frac{\exp(-\alpha r)}{r^2}=\alpha^2\frac{s_2}{(1-s_2)^2},\quad \frac{\exp(-\alpha r)}{2r^3}=\frac{\alpha^3}{2}\frac{s_2}{(1-s_2)^3},\quad \frac{l(l+1)}{2\mu r^4}=\frac{l(l+1)\alpha^4}{2\mu(1-s_2)^4}. \tag{37}$$

This gives the Hamiltonian operators $H_{nc}^{me}(r,\pi,\Theta,\overline{\Phi})$ and $H_{nc}^{ms}(r,\pi,\Theta,\overline{\Phi})$ of the IMMSEPM and IMSSKPM as follows:

$$H_{pert}^{me}(r,\pi,\Theta,\overline{\Phi})=\left(\frac{K_1s_1^{3/2}}{(1-s_1)^3}+\frac{K_2s_1^{5/2}}{(1-s_1)^3}+\frac{K_3s_1^{3/2}}{(1-s_1)^4}+\frac{K_4s_1^{7/2}}{(1-s_1)^4}+\frac{K_5s_1^{5/2}}{(1-s_1)^4}+\frac{K_6s_1^{3/2}}{(1-s_1)^2}+\frac{K_7s_1^2}{(1-s_1)^4}\right)\mathbf{L\Theta}$$
$$+\frac{\mathbf{L}.\overline{\mathbf{\Phi}}}{2\mu}+O\left(\Theta^2,\overline{\Phi}^2\right) \tag{38}$$

and

$$H_{pert}^{ms}(r,\pi,\Theta,\overline{\Phi})=\left(\frac{M_1s_2}{(1-s_2)^3}+\frac{M_2s_2^2}{(1-s_2)^3}+\frac{M_3s_2}{(1-s_2)^4}+\frac{M_4s_2^4}{(1-s_2)^4}+\frac{M_5s_2^2}{(1-s_2)^4}+\frac{M_6s_2}{(1-s_2)^4}+\frac{M_7s_2}{(1-s_2)^2}\right.$$
$$\left.+\frac{M_8s_2}{(1-s_2)^3}+\frac{M_9}{(1-s_2)^4}\right)\mathbf{L}.\mathbf{\Theta}+\frac{\mathbf{L}.\overline{\mathbf{\Phi}}}{2\mu}+O\left(\Theta^2,\overline{\Phi}^2\right) \tag{39}$$

with $K_i=2\alpha T_i(i=\overline{1,6})$, $M_i=\alpha L_j(j=\overline{1,5})$ while $K_7=-\frac{16l(l+1)\alpha^4}{2\mu}$, $M_6=L_6\frac{\alpha^4}{2}$, $M_7=L_7\alpha^2$, $M_8=L_8\frac{\alpha^3}{2}$ and $M_9=-\frac{l(l+1)\alpha^4}{2\mu}$. The MMSEP and the MSSKP models are extended by including new radial terms ($\frac{s_1^{3/2}}{(1-s_1)^3}$, $\frac{s_1^{5/2}}{(1-s_1)^3}$, $\frac{s_1^{3/2}}{(1-s_1)^4}$, $\frac{s_1^{7/2}}{(1-s_1)^4}$, $\frac{s_1^{5/2}}{(1-s_1)^4}$, $\frac{s_1^{5/2}}{(1-s_1)^4}$, $\frac{s_1^{3/2}}{(1-s_1)^2}$ and $\frac{s_1^2}{(1-s_1)^4}$) and ($\frac{s_2}{(1-s_2)^3}$, $\frac{s_2^2}{(1-s_2)^3}$, $\frac{s_2}{(1-s_2)^4}$, $\frac{s_2^4}{(1-s_2)^4}$, $\frac{s_2^2}{(1-s_2)^4}$, $\frac{s_2}{(1-s_2)^4}$, $\frac{s_2}{(1-s_2)^2}$, $\frac{s_2}{(1-s_2)^3}$ and $\frac{1}{(1-s_2)^4}$) to become the IMMSEPM and the IMSSKPM in 3D(NR)-NCPS regimes. The new additive parts of the Hamiltonian operators H_{pert}^{me} and H_{pert}^{ms} are also proportional to the infinitesimal couplings ($\mathbf{L}.\mathbf{\Theta}$ and $\mathbf{L}.\overline{\mathbf{\Phi}}$). This is logical from a physical point of view because it explains the interaction between the physical properties of the studied potential $\mathbf{L}$ and the topological properties resulting from the deformation of phase-space that is described with ($\mathbf{\Theta}$ and $\overline{\mathbf{\Phi}}$). In the 3D(NR)-NCPS regimes, this enables us to consider the new additive parts of the Hamiltonian operators (H_{pert}^{me} and H_{pert}^{ms}) as perturbation operators in comparison with the main potentials H_{me} and H_{ms} (parent Hamiltonian operators for the MMSEP and the MSSKP models). That is, the inequality $H_{pert}^{me}\ll H_{me}$ and $H_{pert}^{ms}\ll H_{ms}$ has been satisfied. In other words, all of the physical arguments justifying the use of the time-independent perturbation theory are achieved. We can provide a complete prescription for finding out the energy level of any generalized excited states as a result.

3.2. Non-relativistic expectation values under the IMMSEPM and the IMSSKPM in 3D(NR)-NCPS regimes

In this subsection, we want to apply the perturbative theory in the case of 3D(NR)-NCPS regimes and we find the non-relativistic expectation values ($\left\langle\frac{s_1^{3/2}}{(1-s_1)^3}\right\rangle_{(nlm)}^{me}\equiv X_{1(nlm)}^{me}$, $\left\langle\frac{s_1^{5/2}}{(1-s_1)^3}\right\rangle_{(nlm)}^{me}\equiv X_{2(nlm)}^{me}$, $\left\langle\frac{s_1^{3/2}}{(1-s_1)^4}\right\rangle_{(nlm)}^{me}\equiv X_{3(nlm)}^{me}$, $\left\langle\frac{s_1^{7/2}}{(1-s_1)^4}\right\rangle_{(nlm)}^{me}\equiv X_{4(nlm)}^{me}$, $\left\langle\frac{s_1^{5/2}}{(1-s_1)^4}\right\rangle_{(nlm)}^{me}\equiv X_{5(nlm)}^{me}$, $\left\langle\frac{s_1^{3/2}}{(1-s_1)^2}\right\rangle_{(nlm)}^{me}\equiv X_{6(nlm)}^{me}$ and $\left\langle\frac{s_1^2}{(1-s_1)^4}\right\rangle_{(nlm)}^{me}\equiv X_{7(nlm)}^{me}$) and ($\left\langle\frac{s_2}{(1-s_2)^3}\right\rangle_{(nlm)}^{ms}\equiv Y_{1(nlm)}^{ms}$, $\left\langle\frac{s_2^2}{(1-s_2)^3}\right\rangle_{(nlm)}^{ms}\equiv Y_{2(nlm)}^{ms}$, $\left\langle\frac{s_2}{(1-s_2)^4}\right\rangle_{(nlm)}^{ms}\equiv Y_{3(nlm)}^{ms}$, $\left\langle\frac{s_2^4}{(1-s_2)^4}\right\rangle_{(nlm)}^{ms}\equiv Y_{4(nlm)}^{ms}$, $\left\langle\frac{s_2^2}{(1-s_2)^4}\right\rangle_{(nlm)}^{ms}\equiv Y_{5(nlm)}^{ms}$, $\left\langle\frac{s_2}{(1-s_2)^4}\right\rangle_{(nlm)}^{ms}\equiv Y_{6(nlm)}^{ms}$, $\left\langle\frac{s_2}{(1-s_2)^2}\right\rangle_{(nlm)}^{ms}\equiv Y_{7(nlm)}^{ms}$, $\left\langle\frac{s_2}{(1-s_2)^3}\right\rangle_{(nlm)}^{ms}\equiv Y_{8(nlm)}^{ms}$ and $\left\langle\frac{1}{(1-s_2)^4}\right\rangle_{(nlm)}^{ms}\equiv Y_{9(nlm)}^{ms}$) for the homogeneous ($K_2$ and H_2) and heterogeneous LiH, HCl, HF, DF, and CsO diatomic molecules, taking into account the unperturbed wave functions $\Psi^{(1)}(r,\Omega_3)$ and $\Psi^{(2)}(r,\Omega_3)$ which we have seen previously in Eqs. (11) and (12) under the IMMSEPM and the IMSSKPM. After straightforward calculations, we obtain the previous expectation values of the IMMSEPM model using the standard perturbation theory in the first order as follows:

$$X_{1(nlm)}^{me}=N_n^2\int_0^{+\infty}s_1^{2\sqrt{-\Lambda_{nl}}+3/2}(1-s_1)^{(\chi_l-2)}W_{me}ds_1 \tag{40}$$

$$X^{me}_{2(nlm)} = N_n^2 \int_0^{+\infty} s_1^{2\sqrt{-\Lambda_{nl}}+5/2}(1-s_1)^{(\chi_l-2)} W_{me} ds_1 \tag{41}$$

$$X^{me}_{3(nlm)} = N_n^2 \int_0^{+\infty} s_1^{2\sqrt{-\Lambda_{nl}}+3/2}(1-s_1)^{(\chi_l-3)} W_{me} ds_1 \tag{42}$$

$$X^{me}_{4(nlm)} = N_n^2 \int_0^{+\infty} s_1^{2\sqrt{-\Lambda_{nl}}+7/2}(1-s_1)^{(\chi_l-3)} W_{me} ds_1 \tag{43}$$

$$X^{me}_{5(nlm)} = N_n^2 \int_0^{+\infty} s_1^{2\sqrt{-\Lambda_{nl}}+5/2}(1-s_1)^{(\chi_l-3)} W_{me} ds_1 \tag{44}$$

$$X^{me}_{6(nlm)} = N_n^2 \int_0^{+\infty} s_1^{2\sqrt{-\Lambda_{nl}}+3/2}(1-s_1)^{(\chi_l-1)} W_{me} ds_1 \tag{45}$$

and

$$X^{me}_{7(nlm)} = N_n^2 \int_0^{+\infty} s_1^{2\sqrt{-\Lambda_{nl}}+2}(1-s_1)^{(\chi_l-3)} W_{me} ds_1 \tag{46}$$

while the expectation values the IMMSEPM:

$$Y^{ms}_{1(nlm)} = A_{nl}^2 \int_0^{+\infty} s_2^{2\Omega_{1nl}+1}(1-s_2)^{(2\Omega_{2l}-2)} W_{ms} ds_2 \tag{47}$$

$$Y^{ms}_{2(nlm)} = A_{nl}^2 \int_0^{+\infty} s_2^{2\Omega_{1nl}+2}(1-s_2)^{(2\Omega_{2l}-2)} W_{ms} ds_2 \tag{48}$$

$$Y^{ms}_{3(nlm)} = A_{nl}^2 \int_0^{+\infty} s_2^{2\Omega_{1nl}+1}(1-s_2)^{(2\Omega_{2l}-3)} W_{ms} ds_2 \tag{49}$$

$$Y^{ms}_{4(nlm)} = A_{nl}^2 \int_0^{+\infty} s_2^{2\Omega_{1nl}+4}(1-s_2)^{(2\Omega_{2l}-3)} W_{ms} ds_2 \tag{50}$$

$$Y^{ms}_{5(nlm)} = A_{nl}^2 \int_0^{+\infty} s_2^{2\Omega_{1nl}+2}(1-s_2)^{(2\Omega_{2l}-3)} W_{ms} ds_2 \tag{51}$$

$$Y^{ms}_{6(nlm)} = A_{nl}^2 \int_0^{+\infty} s_2^{2\Omega_{1nl}+1}(1-s_2)^{(2\Omega_{2l}-3)} W_{ms} ds_2 \tag{52}$$

$$Y^{ms}_{7(nlm)} = A^2_{nl}\int_0^{+\infty} s_2^{2\Omega_{1nl}+1}(1-s_2)^{(2\Omega_{2l}-1)}W_{ms}ds_2 \tag{53}$$

$$Y^{ms}_{8(nlm)} = A^2_{nl}\int_0^{+\infty} s_2^{2\Omega_{1nl}+1}(1-s_2)^{(2\Omega_{2l}-2)}W_{ms}ds_2 \tag{54}$$

and

$$Y^{ms}_{9(nlm)} = A^2_{nl}\int_0^{+\infty} s_2^{2\Omega_{1nl}}(1-s_2)^{(2\Omega_{2l}-3)}W_{ms}ds_2 \tag{55}$$

We have introduced valuable abbreviations $\langle X\rangle^{me/ms}_{(nlm)} = \langle n,l,m|X|n,l,m\rangle$ to avoid the extra burden of writing, X equal to the terms ($\frac{s_1^{3/2}}{(1-s_1)^3}$, $\frac{s_1^{5/2}}{(1-s_1)^3}$, $\frac{s_1^{3/2}}{(1-s_1)^4}$, $\frac{s_1^{7/2}}{(1-s_1)^4}$, $\frac{s_1^{5/2}}{(1-s_1)^4}$, $\frac{s_1^{5/2}}{(1-s_1)^4}$, $\frac{s_1^{3/2}}{(1-s_1)^2}$ and $\frac{s_1^2}{(1-s_1)^4}$) and ($\frac{s_2}{(1-s_2)^3}$, $\frac{s_2^2}{(1-s_2)^3}$, $\frac{s_2}{(1-s_2)^4}$, $\frac{s_2^4}{(1-s_2)^4}$, $\frac{s_2^2}{(1-s_2)^4}$, $\frac{s_2}{(1-s_2)^4}$, $\frac{s_2}{(1-s_2)^2}$, $\frac{s_2}{(1-s_2)^3}$ and $\frac{1}{(1-s_2)^4}$). Introducing the change of variable $s_1 = \exp(-2\alpha r)$ and $s_2 = \exp(-\alpha r)$. This maps the region ($0 \le r \le +\infty \to 0 \le (s_1, s_2) \le 1$) and allows us to obtain dr equal to ($-\frac{ds_1}{2\alpha s_1}$ or $-\frac{ds_2}{\alpha s_2}$) and transform Eq. (40) to Eq. (55) into the following form:

$$X^{me}_{1(nlm)} = \frac{N_n^2}{2\alpha}\int_0^{+1} s_1^{2\sqrt{-\Lambda_{nl}}+3/2-1}(1-s_1)^{(\chi_l-2)}W_{me}ds_1 \tag{56}$$

$$X^{me}_{2(nlm)} \equiv \frac{N_n^2}{2\alpha}\int_0^{+1} s_1^{2\sqrt{-\Lambda_{nl}}+5/2-1}(1-s_1)^{(\chi_l-2)}W_{me}ds_1 \tag{57}$$

$$X^{me}_{3(nlm)} \equiv \frac{N_n^2}{2\alpha}\int_0^{+1} s_1^{2\sqrt{-\Lambda_{nl}}+3/2-1}(1-s_1)^{(\chi_l-3)}W_{me}ds_1 \tag{58}$$

$$X^{me}_{4(nlm)} \equiv \frac{N_n^2}{2\alpha}\int_0^{+1} s_1^{2\sqrt{-\Lambda_{nl}}+7/2-1}(1-s_1)^{(\chi_l-3)}W_{me}ds_1 \tag{59}$$

$$X^{me}_{5(nlm)} \equiv \frac{N_n^2}{2\alpha}\int_0^{+1} s_1^{2\sqrt{-\Lambda_{nl}}+5/2-1}(1-s_1)^{(\chi_l-3)}W_{me}ds_1 \tag{60}$$

$$X^{me}_{6(nlm)} \equiv \frac{N_n^2}{2\alpha}\int_0^{+1} s_1^{2\sqrt{-\Lambda_{nl}}+3/2-1}(1-s_1)^{(\chi_l-1)}W_{me}ds_1 \tag{61}$$

and

$$X^{me}_{7(nlm)} = \frac{N_n^2}{2\alpha}\int_0^{+1} s_1^{2\sqrt{-\Lambda_{nl}}+2-1}(1-s_1)^{(\chi_l-3)}W_{me}ds_1 \tag{62}$$

While the expectation values the IMMSEPM:

$$Y^{ms}_{1(nlm)} = \frac{A^2_{nl}}{\alpha}\int_0^{+1} s_2^{2\Omega_{1nl}+1-1}(1-s_2)^{(2\Omega_{2l}-2)}W_{ms}ds_2 \tag{63}$$

$$Y^{ms}_{2(nlm)} = \frac{A^2_{nl}}{\alpha}\int_0^{+1} s_2^{2\Omega_{1nl}+2-1}(1-s_2)^{(2\Omega_{2l}-2)}W_{ms}ds_2 \tag{64}$$

$$Y^{ms}_{3(nlm)} = \frac{A^2_{nl}}{\alpha}\int_0^{+1} s_2^{2\Omega_{1nl}+1-1}(1-s_2)^{(2\Omega_{2l}-3)}W_{ms}ds_2 \tag{65}$$

$$Y^{ms}_{4(nlm)}\frac{A^2_{nl}}{\alpha}\int_0^{+1} s_2^{2\Omega_{1nl}+4-1}(1-s_2)^{(2\Omega_{2l}-3)}W_{ms}ds_2 \tag{66}$$

$$Y^{ms}_{5(nlm)}\frac{A^2_{nl}}{\alpha}\int_0^{+1} s_2^{2\Omega_{1nl}+2-1}(1-s_2)^{(2\Omega_{2l}-3)}W_{ms}ds_2 \tag{67}$$

$$Y^{ms}_{6(nlm)}\frac{A^2_{nl}}{\alpha}\int_0^{+1} s_2^{2\Omega_{1nl}+1-1}(1-s_2)^{(2\Omega_{2l}-3)}W_{ms}ds_2 \tag{68}$$

$$Y^{ms}_{7(nlm)}\frac{A^2_{nl}}{\alpha}\int_0^{+1} s_2^{2\Omega_{1nl}+1-1}(1-s_2)^{(2\Omega_{2l}-1)}W_{ms}ds_2 \tag{69}$$

$$Y^{ms}_{8(nlm)}\frac{A^2_{nl}}{\alpha}\int_0^{+1} s_2^{2\Omega_{1nl}+1-1}(1-s_2)^{(2\Omega_{2l}-2)}W_{ms}ds_2 \tag{70}$$

and

$$Y^{ms}_{9(nlm)} = \frac{A^2_{nl}}{\alpha}\int_0^{+1} s_2^{2\Omega_{1nl}-1}(1-s_2)^{(2\Omega_{2l}-3)}W_{ms}ds_2 \tag{71}$$

Use the special integral formula in the following Eq. (72) to calculate the integrals in Eq. (56) to Eq. (71) for any generalized excited state [54]:

$$\int_0^{+1} s_1^{2\alpha-1}(1-s_1)^{2(\beta+1)}[2F_1\left(-n, n+2\left(\beta+\alpha+1\right); 2\alpha+1, s_1\right),]^2 ds_1$$

$$= \frac{n!\left(n+\beta+1\right)\Gamma\left(2\alpha\right)\Gamma\left(n+2\left(\beta+1\right)\right)\Gamma\left(2\alpha+1\right)}{\left(n+\beta+1+\alpha\right)\Gamma\left(n+2\alpha+1\right)\Gamma\left(n+2\alpha+2\left(\beta+1\right)\right)} \tag{72}$$

Here, $\Gamma(n+2(\beta+1))$ Stands for the standard Gamma function. The following results can be obtained by utilizing the integrals in Eq. (72), to identify Eq. (56) to Eq. (71):

$$X^{me}_{1(nlm)} \equiv \Xi^1_{me}\frac{\Gamma\left(2\sqrt{-\Lambda_{nl}}+3/2\right)\Gamma\left(n+\chi_l-2\right)\Gamma\left(2\sqrt{-\Lambda_{nl}}+5/2\right)}{\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+5/2\right)\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+\chi_l-1/2\right)} \tag{73}$$

$$X^{me}_{2(nlm)} \equiv \Xi^{2}_{me}\frac{\Gamma\left(2\sqrt{-\Lambda_{nl}}+5/2\right)\Gamma\left(n+\chi_{l}-2\right)\Gamma\left(2\sqrt{-\Lambda_{nl}}+7/2\right)}{\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+7/2\right)\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+\chi_{l}+1/2\right)} \tag{74}$$

$$X^{me}_{3(nlm)} \equiv \Xi^{3}_{me}\frac{\Gamma\left(2\sqrt{-\Lambda_{nl}}+3/2\right)\Gamma\left(n+\chi_{l}-3\right)\Gamma\left(2\sqrt{-\Lambda_{nl}}+5/2\right)}{\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+5/2\right)\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+\chi_{l}-3/2\right)} \tag{75}$$

$$X^{me}_{4(nlm)} \equiv \Xi^{4}_{me}\frac{\Gamma\left(2\sqrt{-\Lambda_{nl}}+7/2\right)\Gamma\left(n+2\left(\nu+1\right)\right)\Gamma\left(2\sqrt{-\Lambda_{nl}}+9/2\right)}{\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+9/2\right)\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+\chi_{l}+1/2\right)} \tag{76}$$

$$X^{me}_{5(nlm)} \equiv \Xi^{5}_{me}\frac{\Gamma\left(2\sqrt{-\Lambda_{nl}}+5/2\right)\Gamma\left(n+\chi_{l}-3\right)\Gamma\left(2\sqrt{-\Lambda_{nl}}+7/2\right)}{\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+7/2\right)\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+\chi_{l}-1/2\right)} \tag{77}$$

$$X^{me}_{6(nlm)} \equiv \Xi^{6}_{me}\frac{\Gamma\left(2\sqrt{-\Lambda_{nl}}+3/2\right)\Gamma\left(n+\chi_{l}-1\right)\Gamma\left(2\sqrt{-\Lambda_{nl}}+5/2\right)}{\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+5/2\right)\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+\chi_{l}+1/2\right)} \tag{78}$$

and

$$X^{me}_{7(nlm)} = \Xi^{7}_{me}\frac{\Gamma\left(2\sqrt{-\Lambda_{nl}}+2\right)\Gamma\left(n+\chi_{l}-3\right)\Gamma\left(2\sqrt{-\Lambda_{nl}}+3\right)}{\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+3\right)\Gamma\left(n+2\sqrt{-\Lambda_{nl}}+\chi_{l}-1\right)} \tag{79}$$

While the expectation values the IMMSEPM model:

$$Y^{ms}_{1(nlm)} = \Xi^{1}_{ms}\frac{\Gamma\left(2\Omega_{1nl}+1\right)\Gamma\left(n+2\Omega_{2l}-2\right)\Gamma\left(2\Omega_{1nl}+2\right)}{\Gamma\left(n+2\Omega_{1nl}+2\right)\Gamma\left(n+2\Omega_{1nl}+2\Omega_{2l}-1\right)} \tag{80}$$

$$Y^{ms}_{2(nlm)} = \Xi^{2}_{ms}\frac{\Gamma\left(2\Omega_{1nl}+2\right)\Gamma\left(n+2\left(\nu+1\right)\right)\Gamma\left(2\Omega_{1nl}+3\right)}{\Gamma\left(n+2\Omega_{1nl}+3\right)\Gamma\left(n+2\Omega_{1nl}+2\Omega_{2l}\right)} \tag{81}$$

$$Y^{ms}_{3(nlm)} = \Xi^{3}_{ms}\frac{\Gamma\left(2\Omega_{1nl}+1\right)\Gamma\left(n+2\Omega_{2l}-3\right)\Gamma\left(2\Omega_{1nl}+2\right)}{\Gamma\left(n+2\Omega_{1nl}+2\right)\Gamma\left(n+2\Omega_{1nl}+2\Omega_{2l}-2\right)} \tag{82}$$

$$Y^{ms}_{4(nlm)} = \Xi^{4}_{ms}\frac{\Gamma\left(2\Omega_{1nl}+4\right)\Gamma\left(n+2\left(\nu+1\right)\right)\Gamma\left(2\Omega_{1nl}+5\right)}{\Gamma\left(n+2\Omega_{1nl}+5\right)\Gamma\left(n+2\Omega_{1nl}+2\Omega_{2l}+1\right)} \tag{83}$$

$$Y^{ms}_{5(nlm)} = \Xi^{5}_{ms}\frac{\Gamma\left(2\Omega_{1nl}+2\right)\Gamma\left(n+2\Omega_{2l}-3\right)\Gamma\left(2\Omega_{1nl}+3\right)}{\Gamma\left(n+2\Omega_{1nl}+3\right)\Gamma\left(n+2\Omega_{1nl}+2\Omega_{2l}-1\right)} \tag{84}$$

$$Y^{ms}_{6(nlm)} = \Xi^{6}_{ms}\frac{\Gamma\left(2\Omega_{1nl}+1\right)\Gamma\left(n+2\Omega_{2l}-3\right)\Gamma\left(2\Omega_{1nl}+2\right)}{\Gamma\left(n+2\Omega_{1nl}+2\right)\Gamma\left(n+2\Omega_{1nl}+2\Omega_{2l}-2\right)} \tag{85}$$

$$Y^{ms}_{7(nlm)} = \Xi^{7}_{ms}\frac{\Gamma\left(2\Omega_{1nl}+1\right)\Gamma\left(n+2\Omega_{2l}-1\right)\Gamma\left(2\Omega_{1nl}+2\right)}{\Gamma\left(n+2\Omega_{1nl}+2\right)\Gamma\left(n+2\Omega_{1nl}+2\Omega_{2l}\right)} \tag{86}$$

$$Y^{ms}_{8(nlm)} = \Xi^{8}_{ms}\frac{\Gamma\left(2\Omega_{1nl}+1\right)\Gamma\left(n+2\Omega_{2l}-2\right)\Gamma\left(2\Omega_{1nl}+2\right)}{\Gamma\left(n+2\Omega_{1nl}+2\right)\Gamma\left(n+2\Omega_{1nl}+2\Omega_{2l}-1\right)} \tag{87}$$

and

$$Y^{ms}_{9(nlm)} = \Xi^{9}_{ms}\frac{\Gamma\left(2\Omega_{1nl}\right)\Gamma\left(n+2\left(\nu+1\right)\right)\Gamma\left(2\Omega_{1nl}+1\right)}{\Gamma\left(n+2\Omega_{1nl}+1\right)\Gamma\left(n+2\Omega_{1nl}+2\Omega_{2l}-3\right)} \tag{88}$$

with

$$\Xi^{1}_{me} = \frac{N_{n}^{2}n!\left(n+\chi_{l}/2-1\right)}{2\alpha\left(n+\chi_{l}/2+\sqrt{-\Lambda_{nl}}-1/4\right)}, \quad \Xi^{2}_{me} = \frac{N_{n}^{2}n!\left(n+\chi_{l}/2-1\right)}{2\alpha\left(n+\chi_{l}/2+\sqrt{-\Lambda_{nl}}+1/4\right)}$$

$$\Xi^{3}_{me}=\frac{N_n^2 n!\,(n+\chi_l/2-3/2)}{2\alpha\left(n+\chi_l/2+\sqrt{-\Lambda_{nl}}+3/4\right)},\quad \Xi^{4}_{me}=\frac{N_n^2 n!\,(n+\chi_l/2-3/2)}{2\alpha\left(n+\chi_l/2+\sqrt{-\Lambda_{nl}}+1/4\right)}$$

$$\Xi^{5}_{me}=\frac{N_n^2 n!\,(n+\chi_l/2-3/2)}{2\alpha\left(n+\chi_l/2-3/2+\sqrt{-\Lambda_{nl}}-1/4\right)},\quad \Xi^{6}_{me}=\frac{N_n^2 n!\,(n+\chi_l/2-1/2)}{2\alpha\left(n+\chi_l/2+\sqrt{-\Lambda_{nl}}+1/4\right)}$$

$$\Xi^{7}_{me}=\frac{N_n^2 n!\,(n+\chi_l/2-3/2)}{2\alpha\left(n+\chi_l/2+\sqrt{-\Lambda_{nl}}-1/2\right)},\quad \Xi^{1}_{ms}=\frac{A_{nl}^2 n!\,(n+\Omega_{2l}-1)}{\alpha\,(n+\Omega_{2l}+\Omega_{1nl}-1/2)},$$

$$\Xi^{2}_{ms}=\frac{A_{nl}^2 n!\,(n+\Omega_{2l}-1)}{\alpha\,(n+\Omega_{2l}+\Omega_{1nl})},\quad \Xi^{3}_{ms}=\frac{A_{nl}^2 n!\,(n+\Omega_{2l}-3/2)}{\alpha\,(n+\Omega_{2l}+\Omega_{1nl}-1/2)},$$

$$\Xi^{4}_{ms}=\frac{A_{nl}^2 n!\,(n+\Omega_{2l}-3/2)}{\alpha\,(n+\Omega_{2l}+\Omega_{1nl}-1/2)},\quad \Xi^{5}_{ms}=\frac{A_{nl}^2 n!\,(n+\Omega_{2l}-3/2)}{\alpha\,(n+\Omega_{2l}+\Omega_{1nl}-1/2)},$$

$$\Xi^{6}_{ms}=\frac{A_{nl}^2 n!\,(n+\Omega_{2l}-3/2)}{\alpha\,(n+\Omega_{2l}+\Omega_{1nl}-1/2)},\quad \Xi^{7}_{ms}=\frac{A_{nl}^2 n!\,(n+\Omega_{2l}-1/2)}{\alpha\,(n+\Omega_{2l}+\Omega_{1nl})},$$

$$\Xi^{8}_{ms}=\frac{A_{nl}^2 n!\,(n+\Omega_{2l}-1)}{\alpha\,(n+\Omega_{2l}+\Omega_{1nl}-1/2)}\quad\text{and}\quad \Xi^{9}_{ms}=\frac{A_{nl}^2 n!\,(n+\Omega_{2l}-3/2)}{\alpha\,(n+\Omega_{2l}+\Omega_{1nl}-3/2)}.$$

This integral relationship in Eq. (72) has been previously applied in many works, including Ahmadov et al. [93] and by Tas et al. [94].

3.3. Impact phase-space deformation on NR energies under the IMMSEPM and the IMSSKPM in 3D(NR)-NCPS symmetries

We devote this partial section to the energy corrections for both potentials under study (the IMMSEPM and the IMSSKPM) in 3D(NR)-NCPS symmetries by exploiting the degree of freedom included in the noncommutative parameters ($\theta_{\mu\nu}$ and $\overline{\Phi}_{\mu\nu}$) and then applying the principle of superposition to achieve a comprehensive correction that considers all partial states. The global effective potentials ($Z_{me}^{nc-eff}(r,\Theta)$ and $Z_{ms}^{nc-eff}(r,\Theta)$) are the sum of three potentials ($Z_{nl}^{me}(r)$, $\frac{l(l+1)}{r^2}$ and $Z_{me}^{pert}(r,\Theta)$) and ($Z_{nl}^{ms}(r)$, $\frac{l(l+1)}{r^2}$ and $Z_{ms}^{pert}(r,\Theta)$) that are responsible for producing total non-relativistic energies within the framework of 3D(NR)-NCPS symmetries. Naturally, the effective potentials ($Z_{nl}^{me}(r)+\frac{l(l+1)}{r^2}$) and ($Z_{nl}^{ms}(r)+\frac{l(l+1)}{r^2}$) are responsible for the non-relativistic energies E_{nl}^{me} and E_{nl}^{ms} SE, known in the literature we have seen in Eq. (14), is dominant in the absence of phase-space deformation. At the same time, the spontaneously generated potentials $V_{me}^{pert}(r,\Theta)$ and $V_{ms}^{pert}(r,\Theta)$ Due to phase-space deformation, it will play the role of the self-sources of corrected non-relativistic energies in 3D(NR)-NCPS symmetries. Considering that the NC-phase-space parameters $\boldsymbol{\Theta}(\theta_{12},\theta_{23},\theta_{13})/2$ and $\overline{\boldsymbol{\Phi}}(\overline{\Phi}_{12},\overline{\Phi}_{23},\overline{\Phi}_{13})$ are arbitrary, we deal with them according to the appropriate physical need. Firstly, the influence of the perturbed spin-orbit can be generated from the perturbed potential $V_{me}^{pert}(r,\Theta)$ corresponding to the homogeneous (K_2 and H_2) and heterogeneous LiH, HCl, HF, DF, and CsO diatomic molecules. We obtain the perturbed spin-orbit effective potentials by replacing the coupling of the angular momentum $\mathbf{L}$ operator and the NC vectors $\boldsymbol{\Theta}(\theta_{12},\theta_{23},\theta_{13})/2$ and $\overline{\boldsymbol{\Phi}}(\overline{\Phi}_{12},\overline{\Phi}_{23},\overline{\Phi}_{13})$ with the new equivalent coupling:

$$\mathbf{L}.\boldsymbol{\Theta},\ \mathbf{L}.\overline{\boldsymbol{\Phi}})\rightarrow\left(\Theta,\overline{\Phi}\right)\mathbf{L}.\mathbf{S} \tag{89}$$

with Θ and $\overline{\Phi}$ are equals to $\sqrt{\theta_{12}^2+\theta_{23}^2+\theta_{13}^2}$ and $\sqrt{\overline{\Phi}_{12}^2+\overline{\Phi}_{23}^2+\overline{\Phi}_{13}^2}$, respectively. We have oriented the spin-s of the homogeneous (K_2 and H_2) and heterogeneous LiH, HCl, HF, DF, and CsO diatomic molecules to become parallels to the vector $\boldsymbol{\Theta}(\theta_{12},\theta_{23},\theta_{13})/2$ and $\overline{\boldsymbol{\Phi}}(\overline{\Phi}_{12},\overline{\Phi}_{23},\overline{\Phi}_{13})$ which interacted with the IMMSEPM and the IMSSKPM models. Additionally, we use the following transformation, which is well-known in 3D(NR)-QM

symmetries:

$$\left(\Theta, \overline{\Phi}\right) \mathbf{L}.\mathbf{S} \to \left(\frac{\Theta}{2}, \frac{\overline{\Phi}}{2}\right) \left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right) \tag{90}$$

In 3D(NR)-NCPS symmetry, the operators ($H_{nc}^{me}(r, \pi, \Theta, \overline{\Phi})$, $H_{nc}^{ms}(r, \pi, \Theta, \overline{\Phi})$, $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J_z}$) form a complete set of conserved physics quantities, and the eigenvalues of the operator $\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2$ are equal to the values $\delta(j, l, s)$ for the homogeneous (K_2 and H_2) and heterogeneous LiH, HCl, HF, DF, and CsO diatomic molecules:

$$\delta\left(j, l, s\right) = \frac{1}{2}\left[j\left(j+1\right) - l\left(l+1\right) - s\left(s+1\right)\right] \tag{91}$$

Here $j \in [|l-s|, |l+s|]$ and $s = \{1/2, 0, 1, ...\}$. As a direct consequence, the partially corrected square energies $\Delta E_{me}^{so}(n, V_0, V_1, V_2, A, B, \alpha, j, l, s)$ and $\Delta E_{ms}^{so}(n, A, B, D_e, r_e, \alpha, j, l, s)$ due to the perturbed effective Hamiltonians $H_{pert}^{me}(r, \pi, \Theta, \overline{\Phi})$ and $H_{pert}^{ms}(r, \pi, \Theta, \overline{\Phi})$ produced for any excited state, in 3D(NR)-NCPS regimes, are as follows:

$$\Delta E_{me}^{so} = \delta\left(j, l, s\right)\left(\Theta \left\langle Z\right\rangle_{(nlm)}^{me} + \frac{\overline{\Phi}}{2\mu}\right) \tag{92}$$

and

$$\Delta E_{ms}^{nr-so} = \delta\left(j, l, s\right)\left(\Theta \left\langle Z\right\rangle_{(nlm)}^{ms} + \frac{\overline{\Phi}}{2\mu}\right) \tag{93}$$

The global non-relativistic expectation values $\left\langle Z\right\rangle_{(nlm)}^{me} \equiv \left\langle Z\right\rangle_{(nlm)}^{me}(n, l, V_0, V_1, V_2, A, B, \alpha)$ and $\left\langle Z\right\rangle_{(nlm)}^{ms} \equiv \left\langle Z\right\rangle_{(nlm)}^{ms}(n, l, A, B, D_e, r_e, \alpha)$ for non-relativistic studied particles, which were created from the effect of the IMMSEPM and the IMSSKPM models, are determined by applying the two expressions, respectively:

$$\left\langle Z\right\rangle_{(nlm)}^{me} = \sum_{i=1}^{7} K^i X_{i(nlm)}^{me} \tag{94}$$

and

$$\left\langle Z\right\rangle_{(nlm)}^{ms} = \sum_{i=1}^{9} M^i Y_{i(nlm)}^{ms} \tag{95}$$

Here $X_{i(nlm)}^{me}$ and $Y_{i(nlm)}^{ms}$ are dependent to $(n, l, V_0, V_1, V_2, A, B, \alpha)$ and $(n, l, A, B, D_e, r_e, \alpha)$, respectively. The second significant physical contribution is due to the action of the magnetic perturbative potential, which causes the effect of the perturbed Hamiltonians $H_{pert}^{me}(r, \pi, \Theta, \overline{\Phi})$ and $H_{pert}^{ms}(r, \pi, \Theta, \overline{\Phi})$ in the 3D(NR)-NCPS symmetries under the IMMSEPM and IMSSKPM. When we replace, we obtain these effective potentials:

$$\left(\mathbf{L}.\boldsymbol{\Theta}, \mathbf{L}.\overline{\boldsymbol{\Phi}}\right) \to \left(\chi \mathbf{L}.\aleph, \overline{\chi}\mathbf{L}.\aleph\right) \tag{96}$$

The condition for the validity of the above relationship requires that the physical units be identical $[\overline{\Phi}] = [\chi][\aleph] \equiv$ length square and $[\eta] = [\overline{\chi}][\aleph] \equiv$ momentum square while ($\aleph$, χ and $\overline{\chi}$) are present, as is the intensity of the magnetic field induced by the effect of the deformation of phase-space geometry and new infinitesimal non-commutativity parameters. This choice comes from the fact that the vectors $\boldsymbol{\Theta}(\theta_{12}, \theta_{23}, \theta_{13})/2$ and $\overline{\boldsymbol{\Phi}}(\overline{\Phi}_{12}, \overline{\Phi}_{23}, \overline{\Phi}_{13})$ are arbitrary. Now, we chose $\aleph$ parallel to the (Oz) axis ($\aleph e_z$) to simplify quantitative calculations without affecting the nature of the physical point of view, and we need to use the following quantum relation:

$$\left\langle n', l', m' \ |L_z|\ n, l, m\right\rangle = m\delta_{n'n}\delta_{l'l}\delta_{m'm}$$

with $m \in [-|l|, +|l|]$ for non-relativistic studied particles. All of these data allow for the discovery of the new square energy shift $\Delta E_{me}^{mg}(n, V_0, V_1, V_2, A, B, \alpha, \chi, m)$ for the homogeneous (K_2 and H_2) and heterogeneous LiH,

HCl, HF, DF, and CsO diatomic molecules due to the perturbed Zeeman effect created by the influence of the IMMSEPM and the IMSSKPM for any excited state in 3D(NR)-NCPS symmetries as follows:

$$\Delta E_{me}^{mg} = \aleph\left(\chi\sum_{i=1}^{7} K^{i} X_{i(nlm)}^{me} + \frac{\overline{\chi}}{2\mu}\right) m \tag{97}$$

and

$$\Delta E_{me}^{mg} = \aleph\left(\chi\sum_{i=1}^{9} M^{i} Y_{i(nlm)}^{ms} + \frac{\overline{\chi}}{2\mu}\right) m \tag{98}$$

After we've finished the first and second steps of energy self-production, we'll look at another critical instance using the IMMSEPM and the IMSSKPM in 3D(NR)-NCPS symmetries. This new physical phenomenon emerges spontaneously as a result of the perturbed Hamiltonians $H_{pert}^{me}(r,\pi,\Theta,\overline{\Phi})$ and $H_{pert}^{ms}(r,\pi,\Theta,\overline{\Phi})$. We analyze homogeneous (K_2 and H_2) and heterogeneous diatomic molecules rotating with angular velocity (LiH, HCl, HF, DF, and CsO). The features of this subjective experience are determined by replacing the arbitrary vectors $\boldsymbol{\Theta}(\theta_{12},\theta_{23},\theta_{13})/2$ and $\overline{\boldsymbol{\Phi}}(\overline{\Phi}_{12},\overline{\Phi}_{23},\overline{\Phi}_{13})$ with $\zeta\boldsymbol{\Omega}$. As a result, we can replace the couplings ($\mathbf{L}.\boldsymbol{\Theta}$ and $\mathbf{L}.\overline{\Phi}$) with ($\zeta\mathbf{L}\boldsymbol{\Omega}$ and $\overline{\zeta}\mathbf{L}\boldsymbol{\Omega}$), respectively, as following forms:

$$\langle Z\rangle_{(nlm)}^{me}(n,V_0,V_1,V_2,A,B,\alpha) = \left(\zeta\sum_{i=1}^{7} K^{i} X_{i(nlm)}^{me} + \frac{\overline{\zeta}}{2\mu}\right)\mathbf{L}\boldsymbol{\Omega} \tag{99}$$

and

$$\langle Z\rangle_{(nlm)}^{ms}(n,A,B,D_e,r_e,\alpha) = \left(\zeta\sum_{i=1}^{9} M^{i} Y_{i(nlm)}^{ms} + \frac{\overline{\zeta}}{2\mu}\right)\mathbf{L}\boldsymbol{\Omega} \tag{100}$$

In this case, (ζ and $\overline{\zeta}$) essentially represents infinitesimal real proportional constants. To simplify the calculations, we picked a rotational velocity $\boldsymbol{\Omega}$ parallel to the (Oz) axis $\boldsymbol{\Omega} = \Omega e_z$. This, of course, does not modify the physical properties of the investigated problem as much as it simplifies the computations. The perturbed created spin-orbit coupling then transforms into the following new physical phenomenon:

$$\langle Z(r)\rangle_{(nlm)}^{me} = \left(\zeta\sum_{i=1}^{7} K^{i} X_{i(nlm)}^{me} + \frac{\overline{\zeta}}{2\mu}\right)\Omega L_z \tag{101}$$

and

$$\langle Z(r)\rangle_{(nlm)}^{me} = \left(\zeta\sum_{i=1}^{9} M^{i} Y_{i(nlm)}^{ms}\Omega + \frac{\overline{\zeta}}{2\mu}\right)\Omega L_z \tag{102}$$

In 3D(NR)-NCPS symmetries, the perturbed Hamiltonians $H_{pert}^{me}(r,\pi,\Theta,\overline{\Phi})$ and $H_{pert}^{ms}(r,\pi,\Theta,\overline{\Phi})$ that are generated automatically by the influence of the IMMSEPM and the IMSSKPM for any excited state, will be a source of additional new corrected energy values: $\Delta E_{me}^{nr-rot}(n,l,V_0,V_1,V_2,A,B,\alpha,\zeta,m)$ and $\Delta E_{ms}^{nr-rot}(n,l,A,B,D_e,r_e,\alpha,\zeta,m)$ of the homogeneous (K_2 and H_2) and heterogeneous (LiH, HCl, HF, DF, and CsO) diatomic molecules:

$$\Delta E_{me}^{rot} = \left(\zeta\sum_{i=1}^{7} K^{i} X_{i(nlm)}^{me}\Omega + \frac{\overline{\zeta}}{2\mu}\right)\Omega m \tag{103}$$

and

$$\Delta E_{ms}^{rot} = \left(\zeta\sum_{i=1}^{9} M^{i} Y_{i(nlm)}^{ms}\Omega + \frac{\overline{\zeta}}{2\mu}\right)\Omega m \tag{104}$$

Notably, Medjedel and Bencheikh in their Ref. [95] explored rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two and three dimensions at zero temperature; however, the rotational term was manually introduced to the Hamiltonian operators in this instance. On the other hand, in our investigation, the rotation operators $\langle Z(r)\rangle_{(nlm)}^{r-me}$ and $\langle Z(r)\rangle_{(nlm)}^{r-ms}$ emerge naturally as a result of phase-space deformation under the IMMSEPM and IMSSKPM. In the symmetries of the 3D(NR)-NCPS regimes, under the IMMSEPM and IMSSKPM, The total non-relativistic new energies $E_{nc}^{me}(n, V_0, V_1, V_2, A, B, \alpha, \overline{\Phi}, \chi, \zeta, j, l, s, m) \equiv E_{nc}^{me}$ and $E_{nc}^{ms}(n, V_0, V_1, V_2, A, B, \alpha, \overline{\Phi}, \chi, \zeta, j, l, s, m) \equiv E_{nc}^{ms}$ corresponds to any generalized *state* for the homogeneous (K_2 and H_2) and heterogeneous LiH, HCl, HF, DF, and CsO diatomic molecules, can be evaluated as follows:

$$E_{nc}^{me} = a - q\left(\frac{b}{n+\sigma} + \frac{n+\sigma}{2}\right)^2 + \sum_{i=1}^{7} K^i X_{i(nlm)}^{me}\left((\chi\aleph + \zeta\Omega)m + \Theta\delta\right) + \frac{\overline{\Phi}\delta + \overline{\chi}\aleph m + \overline{\zeta}\Omega m}{2\mu} \tag{105}$$

and

$$E_{nc}^{ms} = V_0A^2 + \frac{\alpha^2 l\left(l+1\right)}{2\mu} - \frac{\alpha^2}{8\mu}\left(n + \Delta + \frac{Q_2}{n+\Delta}\right)^2 + \sum_{i=1}^{9} M^i Y_{i(nlm)}^{ms}\left((\chi\aleph + \zeta\Omega)m + \Theta\delta\right) + \frac{\overline{\Phi}\delta + \overline{\chi}\aleph m + \overline{\zeta}\Omega m}{2\mu} \tag{106}$$

The first two parts (a plus $-q(\frac{b}{n+\sigma} + \frac{n+\sigma}{2})^2$) and ($V_0A^2 + \frac{\alpha^2 l(l+1)}{2\mu}$ plus $\frac{\alpha^2}{8\mu}(n + \Delta + \frac{Q_2}{n+\Delta})^2$) are non-relativistic energies under the MMSEP and the MSSKP obtained from equations of energy in Eq. (14) while the remaining terms are in Eqs. (105) and (106) represent the resulting correction from the deformation phase-space. It is essential to point out that because we have only used corrections of the first order of infinitesimal noncommutativity parameters (Θ, χ, ζ) and ($\overline{\Phi}, \overline{\chi}, \overline{\zeta}$), perturbation theory cannot be used to find corrections of the second-order ($\Theta^2, \chi^2, \zeta^2$) and ($\overline{\Phi}^2, \overline{\chi}^2, \overline{\zeta}^2$).

4. Investigation of significant particular cases of the IMMSEPM and IMSSKPM in 3D(NR)-NCPS regimes

This section will examine the new bound state eigenvalues of the 3D-DSE derived with the IMMSEPM and IMSSKPM in 3D(NR)-NCPS symmetries. Let us conclude it based on the two total energy Eqs. (105) and (106). By appropriately adjusting the potential parameters (A, B, V_0, V_1 and V_2) and (A, B, D_e, r_e) of the MMSEP and MSSKP models, we are now in the process of treating into 3D(NR)-NCPS regime:

(1) Assume that $V_1 = V_2 = 0$ and $D_e = 0$ in Eq. (2.1) and 52.2) we obtain the modified Mobius square potential $(-V_0(\frac{A+B\exp(-2\alpha r)}{1-\exp(-2\alpha r)})^2)$, which was studied in Ref. [5] and the Mobius square potential $V_0(\frac{A+B\exp(-\alpha r)}{1-\exp(-\alpha r)})^2$ that was studied in Ref. [4]. From Eqs. (105) and (106), we get the NR-relativistic new energies $E_{nc}^{imm}(n, V_0, A, B, \alpha, \Theta, \chi, \zeta, \overline{\Phi}, \overline{\chi}, \overline{\zeta}, j, l, s, m)$ and $E_{nc}^{im}(n, A, B, r_e, \alpha, \Theta, \chi, \zeta, \overline{\Phi}, \overline{\chi}, \overline{\zeta}, j, l, s, m)$ for non-relativistic particles under the improved modified Mobius square potential (IMMSP) and the improved Mobius square potential (IMSP), respectively, corresponding to the generalized excited states in the 3D(NR)-NCPS regime as:

$$E_{nc}^{imm} = -V_0A^2 - \frac{2\alpha^2}{\mu}\left(\frac{\frac{\mu V_0(A+B)}{\alpha^2} + \frac{1}{2} + K_1\left(1 - \frac{1}{2} + K_1\right)}{n + \frac{1}{2} + K} + \frac{n + \frac{1}{2} + K_1}{2}\right)^2 + \sum_{i=1}^{7} K^i X_{i(nlm)}^{imm} N_p^{sp} + N_p^{ph} \tag{107}$$

and

$$E_{nc}^{im} = V_0A^2 + \frac{\alpha^2 l\left(l+1\right)}{2\mu} - \frac{\alpha^2}{8\mu}\left(n + \frac{1}{2} + G_1 + \frac{\frac{2\mu V_0\left(A^2 - B^2\right)}{\alpha^2} + l\left(l+1\right)}{n + \frac{1}{2} + G_1}\right)^2 + \sum_{i=1}^{9} M^i Y_{i(nlm)}^{im} N_p^{sp} + N_p^{ph} \tag{108}$$

The connection between non-commutative space (Θ, χ, ζ) and the physical properties of the studied system $(\Lambda, \aleph, m)$ can be summarized into a new representation N_p^{sp} that given by:

$$N_p^{sp} = \Theta\Lambda + (\chi\aleph + \zeta\Omega)\, m$$

While the connected between non-commutative phase $(\overline{\Phi}, \overline{\chi}, \overline{\zeta})$ and the physical properties of the studied system $(\frac{\Lambda}{2\mu}, \frac{\aleph}{2\mu}\frac{\Omega}{2\mu})$ can be summarized into a new representation N_p^{ph} that given by:

$$N_p^{ph} = \overline{\eta}\frac{\Lambda}{2\mu} + \left(\overline{\chi}\frac{\aleph}{2\mu} + \overline{\zeta}\frac{\Omega}{2\mu}\right) m$$

Here K_1 and G_1 are equal to $\frac{1}{2}\sqrt{1 - \frac{2\mu V_0(A+B)^2}{\alpha^2}}$ and $\sqrt{\frac{2\mu V_0(A+B)^2}{\alpha^2} + (l+1/2)^2}$ respectively. The new non-relativistic energy in Eqs. (107) and (108) agrees as a special case within our Refs. [19, 96]. The corresponding new non-relativistic expectation values $X_{i(nlm)}^{imm}(n, V_0, A, B, \alpha)$ and $Y_{i(nlm)}^{im}$ of the IMMSP and the IMSP come from the following limits:

$$X_{i(nlm)}^{imm}(n, V_0, A, B, \alpha) = \lim_{(V_1,V_2)\to(0,0)} X_{i(nlm)}^{me}(n, V_0, V_1, V_2, A, B, \alpha),$$

and

$$Y_{i(nlm)}^{im}(n, A, B, r_e, \alpha) = \lim_{D_e\to 0} Y_{i(nlm)}^{ms}(n, A, B, D_e, r_e, \alpha). \tag{109}$$

(2) Assume that $V_0 = 0$ and $A = B = 0$ in Eqs. (2.1) and (2.2), we obtain the Eckart potential $(-\frac{V_1\exp(-2\alpha r)}{1-\exp(-2\alpha r)} + \frac{V_2\exp(-2\alpha r)}{(1-\exp(-2\alpha r))^2})$, which was studied in Refs. [97, 98] and the Kratzer potential $(-2D_e(\frac{r_e}{r} - \frac{r_e^2}{2r^2}))$ that was studied in Ref. [96]. From Eqs. (105) and (106), we obtain the non-relativistic new energies $E_{nc}^{ie}(n, V_1, V_2, A, B, \alpha, \overline{\Phi}, \chi, \zeta, j, l, s, m)$ and relativistic particles under the improved Eckart potential (IEP) and the improved Kratzer potential (IKP), respectively, corresponding to the generalized excited states in 3D(NR)-NCPS symmetries as:

$$E_{nc}^{ie} = -\frac{2\alpha^2}{\mu}\left(\frac{\frac{\mu(V_1-V_2)}{2\alpha^2} + \frac{1}{2} + K_3\left(1 - \frac{1}{2} + K_3\right)}{n + \frac{1}{2} + K_3} + \frac{n + \frac{1}{2} + K_3}{2}\right)^2 + \sum_{i=1}^{7} K^i X_{i(nlm)}^{ie} N_p^{sp} + N_p^{ph} \tag{110}$$

and

$$E_{nc}^{ik} = \frac{\alpha^2 l\,(l+1)}{2\mu} - \frac{\alpha^2}{8\mu}\left(n + K_4 + \frac{l\,(l+1)}{n + K_4}\right)^2 + \sum_{i=1}^{9} M^i Y_{i(nlm)}^{ie} N_p^{sp} + N_p^{ph} \tag{111}$$

Here K_3 and K_4 are equal to $\frac{1}{2}\sqrt{1 + \frac{2\mu V_2}{\alpha^2}}$ and $(\frac{1}{2} + \sqrt{(l+1/2)^2})$, respectively. The new non-relativistic energy in Eqs. (110) and (111) agrees as a special case within our Refs. [20, 21, 24, 50]. The corresponding new non-relativistic expectation values $X_{i(nlm)}^{ie}(n, V_1, V_2, \alpha)$ and $Y_{i(nlm)}^{ie}(n, D_e, r_e)$ of the IEP and the IKP come from the following limits:

$$(X_{i(nlm)}^{ie}, Y_{i(nlm)}^{ie}) = \left(\lim_{(V_0,A,B)\to(0,0,0)} X_{i(nlm)}^{me}, \lim_{(A,B,,\alpha)\to(0,0,0)} Y_{i(nlm)}^{ms}\right) \tag{112}$$

(3) Assume that $V_0 = V_2 = 0$ in Eq. (2.1) we obtain the Hulthén potential $(\frac{V_1\exp(-2\alpha r)}{1-\exp(-2\alpha r)})$, which was studied in Ref. [99] and other papers. From Eq. (105), we obtain the non-relativistic new energies $E_{nc}^{ih}(n, V_1, \alpha, \overline{\Phi}, \chi, \zeta, j, l, s, m)$ for non-relativistic particles under the improved Hulthén potential (IHP), corresponding to the generalized excited states in 3D(NR)-NCPS symmetries as:

$$E_{nc}^{ie} = -\frac{2\alpha^2}{\mu}\left(\frac{\frac{\mu V}{2\alpha^2} - (n+l+1)}{2\,(n+l+1)}\right)^2 + \sum_{i=1}^{7} K^i X_{i(nlm)}^{ih} N_p^{sp} + N_p^{ph} \tag{113}$$

The obtained value of NR-relativistic energy in Eq. (113) agrees as a special case within our Refs. [48, 100, 101]. The corresponding new non-relativistic expectation values $X^{ih}_{i(nlm)}(n, V_1, \alpha)$ of the IHP come from the following limits:

$$X^{ih}_{i(nlm)} = \lim_{(V_0,A,B)\to(0,0,0)} X^{me}_{i(nlm)} \tag{114}$$

(4) Assume that $V_0 = V_1 = 0$ in Eq. (2.1) yields the Poschl-Teller potential of the form $\frac{V_2 \exp(-2\alpha r)}{1-\exp(-2\alpha r)}$, which was studied in Ref. [102], and other papers. From Eq. (105), we obtain the non-relativistic new energies $E^{ip}_{nc}(n, V_2, A, B, \alpha, \overline{\Phi}, \chi, \zeta, j, l, s, m)$ for non-relativistic particles under the improved Poschl-Teller potential, corresponding to the generalized excited states in 3D(NR)-NCPS symmetries as:

$$E^{ip}_{nc} = -\frac{2\alpha^2}{\mu}\left(\frac{\frac{-\mu V_2}{2\alpha^2} + K_5\left(\frac{1}{2} - K_5\right)}{n + K_5} + \frac{n + K_5}{2}\right)^2 + \sum_{i=1}^{7} K^i X^p_{i(nlm)} N^{sp}_p + N^{ph}_p \tag{115}$$

Here $K_5$is equal to $(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2\mu V_2}{\alpha^2}})$. The corresponding new non-relativistic expectation values $X^{p}_{i(nlm)}(n, V_2, A, B, \alpha)$ of the improved Poschl-Teller potential come from the following limits:

$$X^{p}_{i(nlm)} = \lim_{(V_0,V_1)\to(0,0)} X^{me}_{i(nlm)} \tag{116}$$

5. Study of composite systems under the IMMSEPM and IMSSKPM

This section is devoted to investigating composite systems such as molecules consisting of $N = 2$ particles of masses $m_n (n = 1, 2)$ label the particle under the IMMSEPM and IMSSKPM in 3D(NR)-NCPS regimes. It was discovered that different non-commutative parameters define composite systems of varying masses $q^{(s,h,i)}_{\mu}$ and $\pi^{(s,h,i)}_{\mu}$ as follows [41, 46]:

$$\left[q^s_\mu, q^s_\nu\right]_* = \left[q^h_\mu(t), q^h_\nu(t)\right]_* = \left[q^i_\mu(t), q^i_\nu(t)\right]_* = i\theta^c_{\mu\nu},$$

and

$$\left[\pi^s_\mu, \pi^s_\nu\right]_* = \left[\pi^h_\mu(t), \pi^h_\nu(t)\right]_* = \left[\pi^i_\mu(t), \pi^i_\nu(t)\right]_* = i\overline{\Phi}^c_{\mu\nu}. \tag{117}$$

The non-commutativity parameters are:

$$\left(\theta^c_{\mu\nu}, \overline{\Phi}^c_{\mu\nu}, \alpha_n\right) = \left(\sum_{n=1}^{2}\alpha_n^2\theta^{(n)}_{\mu\nu}, \sum_{n=1}^{2}\alpha_n^2\overline{\Phi}^{(n)}_{\mu\nu}, \frac{m_n}{\sum_n m_n}\right)$$

here $(\theta^{(n)}_{\alpha\beta}, \overline{\Phi}^{(n)}_{\alpha\beta})$ are the non-commutativity parameters for the particle with mass m_n. Note that in the case of a system of two particles with the same mass $m_1 = m_2$ such as the homogeneous (K_2 and H_2) diatomic molecules the parameters $\theta^{(n)}_{\mu\nu} = \theta_{\mu\nu}$ and $\overline{\Phi}^{(n)}_{\mu\nu} = \overline{\Phi}_{\mu\nu}$. Thus, the parameters ($\Theta$, χ, and ζ) and ($\overline{\Phi}$, $\overline{\chi}$ and $\overline{\zeta}$) that appear in Eqs. (105) and (106) are becomes as follows:

$$\Lambda^{c2} = \left(\sum_{n=1}^{2}\alpha_n^2\Lambda^{(n)}_{12}\right)^2 + \left(\sum_{n=1}^{2}\alpha_n^2\Lambda^{(n)}_{23}\right)^2 + \left(\sum_{n=1}^{2}\alpha_n^2\Lambda^{(n)}_{13}\right)^2 \tag{118}$$

with Λ^{c2} can take all roles of parameters ($\Theta^{c2}/\overline{\Phi}^{c2}$, $\chi^{c2}/\overline{\chi}^{c2}$ and $\zeta^{c2}/\overline{\zeta}^{c2}$). In the case of a system of two particles with the same mass $m_1 = m_2$ such as the homogeneous (K_2 and H_2) diatomic molecules:

$$\left(\theta^{(n)}_{\mu\nu}, \overline{\Phi}^{(n)}_{\mu\nu}, \chi^{(n)}_{\mu\nu}, \overline{\chi}^{(n)}_{\mu\nu}, \zeta^{(n)}_{\mu\nu}, \overline{\zeta}^{(n)}_{\mu\nu}\right) = \left(\theta_{\mu\nu}, \overline{\Phi}_{\mu\nu}, \chi_{\mu\nu}, \overline{\chi}_{\mu\nu}, \zeta_{\mu\nu}, \overline{\zeta}_{\mu\nu}\right)$$

We conclude this section by giving the total energy values $E_{nc}^{me}(n, V_0, V_1, V_2, A, B, \alpha, \Theta^c/\overline{\Phi}^c, \chi^c/\overline{\chi}^c, \zeta^c/\overline{\zeta}^c, j, l, s, m) \equiv E_{nc}^{me}$ and $E_{nc}^{ms}(n, A, B, D_e, r_e, \alpha, \Theta^c/\overline{\Phi}^c, \chi^c/\overline{\chi}^c, \zeta^c/\overline{\zeta}^c, j, l, s, m) \equiv E_{nc}^{ms}$ under the IMMSEPM and IMSSKPM and considering those composite systems with different masses are described with different noncommutative parameters for the heterogeneous diatomic LiH, HCl, HF, DF, and CsO as:

$$E_{nc}^{me} = a - q\left(\frac{b}{n+\sigma} + \frac{n+\sigma}{2}\right)^2 + \sum_{i=1}^{7} K^i X_{i(nlm)}^{me}\left(N_p^{sp}\right)^c + N_p^{ph} \tag{119}$$

and

$$E_{nc}^{ms} = V_0 A^2 + \frac{\alpha^2 l\,(l+1)}{2\mu} - \frac{\alpha^2}{8\mu}\left(n + \Delta + \frac{Q_2}{n+\Delta}\right)^2 + \sum_{i=1}^{9} M^i Y_{i(nlm)}^{ms}\left(N_p^{sp}\right)^c + \left(N_p^{ph}\right)^c \tag{120}$$

With $(N_p^{sp})^c$ and $(N_p^{ph})^c$ are determined by:

$$\left(N_p^{sp}\right)^c, \left(N_p^{ph}\right)^c = \left(\Theta^c \Lambda + \left(\chi^c \aleph + \zeta^c \Omega\right) m,\ \overline{\eta}^c \frac{\Lambda}{2\mu} + \left(\overline{\chi}^c \frac{\aleph}{2\mu} + \overline{\zeta}^c \frac{\Omega}{2\mu}\right)\right.$$

6. Thermodynamic quantities of the IMMSEPM and the IMSSKPM in 3D(NR)-NCPS symmetries

The thermodynamic properties of the IMMSEPM and IMSSKPM in 3D(NR)-NCPS symmetries are investigated in this section. The first step toward achieving this goal is to determine its partition function, because it may be used to calculate other thermal parameters such as internal energy, entropy, free energy, and specific heat capacity. The partition function can be computed via direct summation over all potential energy levels at a constant temperature T [103–114]:

$$Z_{me}^{nl} = \sum_{n=0}^{\nu} \exp\left(-\beta E_{nl}^{me}\right) \Rightarrow Z_{me}^{nc} = \sum_{n=0}^{\nu_{max}} \exp\left(-\beta E_{nc}^{me}\right) \tag{121}$$

And a similar equation with the other new partition function Z_{ms}^{nc}. Here $Z_{me}^{nl}\left(n, V_0, V_1, V_2, A, B, \beta, \nu, l\right)$, $Z_{ms}^{nl}(n, A, B, D_e, r_e, \beta, \lambda, l)$, $Z_{me}^{nc}(V_0, V_1, V_2, A, B, \beta, l, \Theta^c/\overline{\Phi}^c, \chi^c/\overline{\chi}^c, \zeta^c/\overline{\zeta}^c)$ and $Z_{ms}^{nc}(n, A, B, D_e, r_e, \beta, l, \Theta^c/\overline{\Phi}^c, \chi^c/\overline{\chi}^c, \zeta^c/\overline{\zeta}^c)$ are the partition functions of (the MMSEP and the MSSKP) and (the IMMSEPM and the IMSSKPM) while or and $(\nu/\lambda, \nu_{max}/\lambda_{max})$ are the upper bound vibration quantum numbers (the maximum quantum numbers) in 3D(NR)-QM and 3D(NR)-NCPS symmetries, respectively. While $\beta = \frac{1}{KT}$, K is the Boltzmann constant. From the beginning of this section, we assumed that the new partition functions (Z_{me}^{nc} and Z_{ms}^{nc}) are dependent on non-commutativity parameters ($\Theta^c/\overline{\Phi}^c$, $\chi^c/\overline{\chi}^c$ and $\zeta^c/\overline{\zeta}^c$) because the corresponding non-relativistic energy we found in these symmetries is related to these parameters. We obtain the parameters $✠_{max}$ and λ_{max} in 3D(NR)-NCPS as a function of corresponding values ν and λ in 3D(NR)-QM as follows:

$$\nu = \left.\frac{dE_{nl}^{me}}{dn}\right]_{n=\nu} = 0 \Rightarrow \left.\frac{dE_{nc}^{me}}{dn}\right]_{n=\nu_{max}} = 0$$

$$\Rightarrow \nu_{max} = \nu + \eta_{per}^{me},$$

and

$$\lambda = \left.\frac{dE_{nl}^{ms}}{dn}\right]_{n=\nu} = 0 \Rightarrow \left.\frac{dE_{nc}^{ms}}{dn}\right]_{n=\lambda_{max}} = 0$$

$$\Rightarrow \lambda_{max} = \lambda + \eta_{per}^{ms} \tag{122}$$

with

$$\eta_{per}^{me}=\frac{d}{dn}\left[\sum_{i=1}^{7}K^{i}X_{i(nlm)}^{me}\left(\left(\chi\aleph+\zeta\Omega\right)m+\Theta\delta\right)+\frac{\overline{\Phi}\delta+\overline{\chi}\aleph m+\overline{\zeta}\Omega m}{2\mu}\right]\Bigg]_{n=\nu_{max}} \tag{123.1}$$

and

$$\eta_{per}^{ms}=\frac{d}{dn}\left[\sum_{i=1}^{9}M^{i}Y_{i(nlm)}^{ms}\left(\left(\chi\aleph+\zeta\Omega\right)m+\Theta\delta\right)+\frac{\overline{\Phi}\delta+\overline{\chi}\aleph m+\overline{\zeta}\Omega m}{2\mu}\right]\Bigg]_{n=\nu_{max}} \tag{123.2}$$

The improved non-relativistic energy E_{nc}^{me} and E_{nc}^{ms} in 3D(NR)-NCPS symmetries in Eqs. (119) and (120) can be simplified for the case $l\neq 0$ as:

$$\left(E_{nc}^{me},E_{nc}^{ms}\right)=\left(E_{nl}^{nr}+\Delta E_{nc}^{me},E_{nl}^{nr}+\Delta E_{nc}^{ms}\right) \tag{124}$$

With

$$\Delta E_{nc}^{me}=\sum_{i=1}^{7}K^{i}X_{i(nlm)}^{me}\left(\left(\chi\aleph+\zeta\Omega\right)m+\Theta\delta\right)+\frac{\overline{\Phi}\delta+\overline{\chi}\aleph m+\overline{\zeta}\Omega m}{2\mu} \tag{125}$$

And a similar equation with the ΔE_{nc}^{ms}. At high temperatures in the classical limit, the modified partition function (Z_{me}^{nc} and Z_{ms}^{nc}) of the IMMSEPM and the IMSSKPM in the frameworks of 3D(NR)-NCPS symmetry can be represented by an integral:

$$Z_{me}^{nc}=\sum_{n=0}^{\nu_{max}}\exp\left(-\beta E_{nl}^{nr}\right)\Rightarrow Z_{me}^{nc}=\int_{0}^{\nu_{max}}\exp\left(-\beta c\right)dn \tag{126}$$

And a similar equation with the NPF Z_{ms}^{nc} obtained from the NPF Z_{me}^{nc} by making $E_{nl}^{nr}\rightarrow E_{nc}^{ms}$. Considering that the two corresponding corrective energy values ΔE_{nc}^{me} and ΔE_{nc}^{ms} for the IMMSEPM and the IMSSKPM, respectively, their contribution is very small compared to the two main terms E_{nl}^{me} and E_{nl}^{ms}. The following approximations are considered physically acceptable:

$$\exp\left(-\beta E_{nc}^{me}\right)=\exp\left(-\beta E_{nr}^{me}\right)-\beta\Delta E_{nc}^{me}\exp\left(-\beta E_{nr}^{me}\right) \tag{127}$$

A similar equation with $\exp(-\beta E_{nc}^{ms})$. Which gives:

$$Z_{me}^{nc}\cong Z_{me}^{nl}\left(n,V_{0},V_{1},V_{2},A,B,\beta,\lambda,l\right)\left[1-\beta\Delta E_{nc}^{me}\right] \tag{128}$$

And

$$Z_{ms}^{nc}\cong Z_{ms}^{nl}\left(n,A,B,D_{e},r_{e},\beta,\lambda,l\right)\left[1-\beta\Delta E_{ns}^{me}\right] \tag{129}$$

Taking into account previous physical considerations, we only accept, in the first place, generally, the terms that are proportional to the infinitesimal NC-parameters ($\Theta^{c}/\overline{\Phi}^{c}$, $\chi^{c}/\overline{\chi}^{c}$ and $\zeta^{c}/\overline{\zeta}^{c}$), Z_{me}^{nc} and Z_{ms}^{nc} are the modified partition functions in 3D(NR)-NCPS symmetry. A rough rewrite of ($\Theta^{c}/\overline{\Phi}^{c}$, $\chi^{c}/\overline{\chi}^{c}$ and $\zeta^{c}/\overline{\zeta}^{c}$) would be as follows:

$$Z_{me}^{nc}\cong Z_{me}^{nl}-\beta\left(\sum_{i=1}^{7}K^{i}X_{i(nlm)}^{me}N_{p}^{sp}+N_{p}^{ph}\right)Z_{me}^{nr}\left(\beta,\nu,l\right) \tag{130}$$

and

$$Z_{ms}^{nc}\cong Z_{ms}^{nl}-\beta\left(\sum_{i=1}^{9}M^{i}Y_{i(nlm)}^{ms}N_{p}^{sp}+N_{p}^{ph}\right)Z_{ms}^{nr}\left(\beta,\lambda,l\right) \tag{131}$$

Here Z_{me}^{nl} are Z_{ms}^{nl} are determined from [12, 13]:

$$Z_{me}^{nl} = \exp\left(-\beta\left(a - qk_1^2\right)\right) - \exp\left(-\beta\left(a - qk_2^2\right)\right) + \exp\left(\beta M_3\rho^2 - \beta M_1\right)\sqrt{\beta M_2}$$

$$\left(\frac{2\upsilon\exp\left(-\frac{\beta M_2}{\upsilon^2}\right)}{\sqrt{\beta M_2}} - 2\sqrt{\pi} + \frac{2\sqrt{\beta M_2}\sqrt{\pi}erf\left(\frac{\beta M_2}{\upsilon}\right)}{\sqrt{\beta M_2}}\right) \tag{132}$$

and

$$Z_{ms}^{nl} = \frac{\exp\left(-\beta\Lambda_0\right)\sqrt{\pi}D^1}{4\sqrt{-\beta Q_1}} - \frac{\exp\left(-\beta\Lambda_0\right)\sqrt{\pi}D^2}{4\sqrt{-\beta Q_1}} \tag{133}$$

Where

$$\rho = n + \sigma, \quad M_1 = \frac{\alpha^2 b^2}{\mu} - a, \quad M_2 = \frac{\alpha^2 b^2}{2\mu}, \quad M_3 = \frac{\alpha^2}{\mu},$$

$$k_1 = \frac{b}{2\sigma} + \frac{\sigma}{2}, k_2 = \frac{b}{2\left(n + \upsilon + \sigma + 1\right)} + \frac{n + \upsilon + \sigma + 1}{2},$$

$$\Lambda_1 = \frac{\sqrt{-Q_1}\left(\lambda^2 + 2\lambda\Delta + \Delta^2 \pm Q_2\right)}{\Delta + \sigma}, \quad \Lambda_2 = \frac{\sqrt{-Q_1}\left(\lambda^2 + 2\lambda\Delta + \Delta^2 - Q_2\right)}{\Delta + \sigma}, \quad \Lambda_3 = \frac{\sqrt{-Q_1}\left(\Delta^2 + Q_2\right)}{\Delta + \sigma},$$

$$\Lambda_4 = \frac{\sqrt{-Q_1}\left(\Delta^2 - Q_2\right)}{\Delta + \sigma}.$$

while

$$D^1 = \exp\left(2\beta Q_1 Q_2\right) erf\left(\Lambda_3\sqrt{\beta}\right) + \exp\left(-2\beta Q_1 Q_2\right) erf\left(\Lambda_4\sqrt{\beta}\right),$$

$$D^2 = \exp\left(2\beta Q_1 Q_2\right) erf\left(\Lambda_1\sqrt{\beta}\right) + \exp\left(-2\beta Q_1 Q_2\right) erf\left(\Lambda_2\sqrt{\beta}\right) \tag{134}$$

Here $erf(z)$ is the error function that is equal to $\frac{2}{\pi}\int_0^z \exp(-t^2 dt)$. To simplify writing the various expressions, we will use the abbreviation $(\Theta/\Phi, \chi/\overline{\chi}, \zeta/\overline{\zeta}) = \{X_{nc}^p\}$. Using the modified partition functions Z_{me}^{nc} and Z_{ms}^{nc} in Eq. (130) Eq. (131), we will see the effect of the deformation in phase-space on thermodynamic values of the improved modified Mobius square plus Eckart potential model and the improved Mobius square plus screened-Kratzer potential model such as new mean energy $(U_{nc}^{me}(n, A, B, D_e, r_e, \beta, l, \{X_{nc}^p\})$ and $U_{nc}^{ms}(n, A, B, D_e, r_e, \beta, l, \{X_{nc}^p\}))$, new free energy $(F_{nc}^{me}(n, V_0, V_1, V_2, A, B, \beta, l, \{X_{nc}^p\})$ and $F_{nc}^{ms}(n, A, B, D_e, r_e, \beta, l, \{X_{nc}^p\}))$ and new entropy $(S_{nc}^{me}(n, V_0, V_1, V_2, A, B, \beta, l, \{X_{nc}^p\})$ and $S_{nc}^{ms}(n, A, B, D_e, r_e, \beta, l, \{X_{nc}^p\}))$. Let's start with the study of new mean energy $U_{nc}^{me}(n, A, B, D_e, r_e, \beta, l, \{X_{nc}^p\})$, which is the quantity required to prepare or improve the system in its internal condition. The effect of the deformation of phase-space on mean energy $U_{me}^{nl}(n, V_0, V_1, V_2, A, B, \beta, \lambda, l)$ and $U_{ms}^{nl}(n, A, B, D_e, r_e, \beta, \lambda, l)$ for the IMMSEPM and the IMSSKPM models are determined by applying the following formula:

$$\Delta U_{me/ms}^{nc} \equiv U_{nc}^{me/ms}\left(\beta, l, \left\{X_{nc}^p\right\}\right) - U_{me/ms}^{nl}\left(\beta, \upsilon/\lambda, l\right) = -\frac{\partial}{\partial\beta}\left[\ln Z_{me/ms}^{nc} - \ln Z_{me/ms}^{nl}\right] \tag{135}$$

A simple calculation gives the effect of the deformation of phase-space on mean energy under the IMMSEPM and the IMSSKPM in 3D(NR)-NCPS symmetries:

$$\Delta U_{nc}^{me} = \frac{\sum_{i=1}^{7} K^i X_{i(nlm)}^{me} N_p^{sp} + N_p^{ph}}{1 - \beta\left(\sum_{i=1}^{7} K^i X_{i(nlm)}^{me} N_p^{sp} + N_p^{ph}\right)} \tag{136}$$

and

$$\Delta U_{nc}^{ms}=\frac{\sum_{i=1}^{9}M^{i}Y_{i(nlm)}^{ms}N_{p}^{sp}+N_{p}^{ph}}{1-\beta\left(\sum_{i=1}^{9}M^{i}Y_{i(nlm)}^{ms}N_{p}^{sp}+N_{p}^{ph}\right)} \tag{137}$$

Thus, in 3D(NR)-NCPS symmetries, the new mean energy $U_{nc}^{me}(n,V_0,V_1,V_2,A,B,\beta,l,\{X_{nc}^{p}\})$ and $U_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$ for the IMMSEPM and the IMSSKPM are equal to the corresponding values in the literature $U_{me}^{nl}(n,V_0,V_1,V_2,A,B,\beta,\lambda,l)$ and $U_{ms}^{nl}(\beta,\lambda,l)$ plus the effect of the deformation of phase-space on it as follows:

$$U_{nc}^{me}=U_{me}^{nl}\left(\beta,\lambda,l\right)+\frac{\sum_{i=1}^{7}K^{i}X_{i(nlm)}^{me}N_{p}^{sp}+N_{p}^{ph}}{1-\beta\left(\sum_{i=1}^{7}K^{i}X_{i(nlm)}^{me}N_{p}^{sp}+N_{p}^{ph}\right)} \tag{138}$$

And a similar equation for U_{nc}^{ms}. Here U_{me}^{nl} and $U_{ms}^{nl}(\beta,\lambda,l)$ are the mean energy for a modified Mobius square plus Eckart potential model and the Mobius square plus screened-Kratzer potential model in 3D(NR)-QM symmetries [12, 13]. Now let's get to the effect of the deformation of phase-space on the free energy $F_{me}^{nl}(\beta,\lambda,l)$ and $F_{ms}^{nl}(\beta,\lambda,l)$ of the modified Mobius square plus Eckart potential model and Mobius square plus screened-Kratzer potential model, which is obtained by applying:

$$\Delta F_{nl}^{me/ms}\equiv F_{nc}^{me/\,ms}-F_{me/ms}^{nl}=-\frac{1}{\beta}\ln Z_{me/ms}^{nc}-\left(-\frac{1}{\beta}\ln Z_{me/ms}^{nr}\right) \tag{139}$$

The impact of phase-space deformation on the free energy $\Delta F_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta F_{nl}^{ms}(\beta,l,\{X_{nc}^{p}\})$ of the IMMSEPM and IMSSKPM, respectively, may be derived as follows:

$$F_{nc}^{me}=F_{me}^{nl}-\frac{1}{\beta}\ln\left[1-\beta\left(\sum_{i=1}^{7}K^{i}X_{i(nlm)}^{me}N_{p}^{sp}+N_{p}^{ph}\right)\right] \tag{140}$$

And a similar equation for ΔF_{nl}^{ms}. Thus, in 3D(NR)-NCPS symmetries, the new free energy $F_{nc}^{me}(n,V_0,V_1,V_2,A,B,\beta,l,\{X_{nc}^{p}\})$ and $F_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$ of the IMMSEPM and the IMSSKPM models, are equal to the corresponding values in the literature $F_{me}^{nl}(n,V_0,V_1,V_2,A,B,\beta,\lambda,l)$ and $F_{ms}^{nl}(\beta,\lambda,l)$ (See Refs. [12, 13]), in 3D(NR)-QM symmetries, plus the effect of the deformation of phase-space on it, respectively as follows:

$$\Delta F_{nl}^{me}\equiv-\frac{1}{\beta}\ln\left[1-\beta\left(\sum_{i=1}^{7}K^{i}X_{i(nlm)}^{me}N_{p}^{sp}+N_{p}^{ph}\right)\right] \tag{141}$$

And a similar equation for F_{nc}^{ms}. The effect of the deformation of phase-space on the specific heat capacity C_{nc}^{me} and C_{nc}^{ms} of the IMMSEPM and the IMSSKP, which is obtained by applying:

$$\Delta C_{nl}^{me/s}\equiv C_{nc}^{me/\mathrm{ms}}-C_{me/ms}^{nl}=-k\beta^{2}\frac{\partial\Delta U_{nc}^{me/ms}}{\partial\beta} \tag{142}$$

The following straightforward calculation shows how the phase-space distortion affects the free energy of the IMSSKPM and IMMSEPM:

$$\Delta C_{nl}^{me}=-k\beta^{2}\frac{\left(\sum_{i=1}^{7}K^{i}X_{i(nlm)}^{me}N_{p}^{sp}+N_{p}^{ph}\right)^{2}}{1-\beta\left(\sum_{i=1}^{7}K^{i}X_{i(nlm)}^{me}N_{p}^{sp}+N_{p}^{ph}\right)} \tag{143}$$

And a similar equation for ΔC_{nl}^{ms}. Within the context of our proposed approximations, this impact can be overlooked, as it is limited to the first order only for the values ($\Theta/\overline{\Phi}$, $\chi/\overline{\chi}$ and $\zeta/\overline{\zeta}$). We analyze the impact of phase-space deformation on the entropies $S_{nc}^{me}(n,V_0,V_1,V_2,A,B,\beta,\nu,l)$ and $S_{nc}^{ms}(n,A,B,D_e,r_e,\beta,\lambda,l)$ that are

derived by using the following expression:

$$\Delta S_{nc}^{me/\mathrm{ms}} \equiv S_{nc}^{me/ms} - S_{me/ms}^{nl} = k\beta^2 \frac{\partial \Delta F_{nc}^{me}}{\partial \beta} \tag{144}$$

The impact of phase-space deformation on the entropy of the IMMSEPM and IMSSKPM can be evaluated simply as follows:

$$\Delta S_{nc}^{me} \equiv k\beta \frac{\sum_{i=1}^{7} K^i X_{i(nlm)}^{me} N_p^{sp} + N_p^{ph}}{1 - \beta\left(\sum_{i=1}^{7} K^i X_{i(nlm)}^{me} N_p^{sp} + N_p^{ph}\right)} \tag{145}$$

And a similar equation for ΔS_{nc}^{ms}. Thus, in 3D(NR)-NCPS symmetries, the new entropy $S_{nc}^{me}(n, V_0, V_1, V_2, A, B, \beta, l, \{\mathrm{X}_{nc}^{p}\})$ and $S_{nc}^{ms}(n, A, B, D_e, r_e, \beta, l, \{\mathrm{X}_{nc}^{p}\})$ of the IMMSEPM and the IMSSKPM are equal to the corresponding values $S_{me}^{nl}(n, V_0, V_1, V_2, A, B, \beta, \lambda, l)$ and $S_{ms}^{nl}(n, A, B, D_e, r_e, \beta, \lambda, l)$ [12, 13] in 3D(NR)-QM plus the effect of the deformation of phase-space on the modified Mobius square plus Eckart potential model and the Mobius square plus screened-Kratzer potential model as follows:

$$S_{nc}^{me} = S_{me}^{nl}(\beta, \lambda, l) + k\beta \frac{\sum_{i=1}^{7} K^i X_{i(nlm)}^{me} N_p^{sp} + N_p^{ph}}{1 - \beta\left(\sum_{i=1}^{7} K^i X_{i(nlm)}^{me} N_p^{sp} + N_p^{ph}\right)} \tag{146}$$

And a similar equation for S_{nc}^{ms}. If the deformation of phase-space effect vanishes when the simultaneous limits $(\Theta/\overline{\Phi}, \chi/\overline{\chi}, \zeta/\overline{\zeta}) \to (0, 0, 0)$ is satisfied, the additive thermodynamic parts $\Delta Z_{nl}^{me/ms}$, $\Delta U_{nl}^{me/ms}$, $\Delta F_{nl}^{me/ms}$, $\Delta S_{nl}^{me/ms}$ and $\Delta C_{nl}^{me/ms}$ naturally also vanish:

$$\lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} \Delta Z_{nl}^{me} = 0 \Leftrightarrow \lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} Z_{nc}^{me} = Z_{nl}^{me}$$

$$\lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} \Delta U_{nl}^{me} = 0 \Leftrightarrow \lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} U_{nc}^{me} = U_{nl}^{me}$$

$$\lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} \Delta F_{nl}^{me} = 0 \Leftrightarrow \lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} F_{nc}^{me} = F_{nl}^{me/ms}$$

$$\lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} \Delta S_{nl}^{me} = 0 \Leftrightarrow \lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} S_{nc}^{me} = S_{nl}^{me}$$

and

$$\lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} \Delta C_{nl}^{me} = 0 \Leftrightarrow \lim_{\{\mathrm{X}_{nc}^{p}\}\to 0} C_{nc}^{me} = C_{nl}^{me}$$

And a similar equations for the IMSSKPM model.

7. Concluding remarks

In our current work, we used the generalized Bopp's shifts method and independent time standard perturbation theory in 3D(NR)-NCPS symmetries to study approximate solutions of the deformed Schrödinger equation for the IMMSEPM and the IMSSKPM with the help of Greene-Aldrich approximation to the centrifugal terms $\frac{l(l+1)}{r^2}$ and $\frac{l(l+1)}{r^4}$ for any $l \neq 0$. We obtained bound new state energies and the IMMSEPM and the IMSSKPM in 3D(NR)-NCPS regimes. We found that for the IMMSEPM and the IMSSKPM, respectively, the new non-relativistic energy eigenvalues seem sensitive to quantum numbers (n, j, l, s and m), the mixed potential depths (V_0, V_1, V_2, A and B) and (A, B, D_e and r_e), the screening parameter α, and non-commutativity phase-space parameters $(\Theta/\overline{\Phi}, \chi/\overline{\chi}, \zeta/\overline{\zeta}) = \{\mathrm{X}_{nc}^{p}\})$. In addition, different special cases of the IMMSEPM and the IMSSKPM have been obtained by adjusting the parameters of the potential, such as the improved modified Mobius square potential, the improved Eckart potential, the improved Hulthén potential, and the improved Poschl-Teller potential.

Additionally, we have studied the effect of the deformation of phase-space on the thermodynamic quantities: the induced partition function ($\Delta Z_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta Z_{nc}^{se}(\beta,l,\{X_{nc}^{p}\})$, the induced mean energy $\Delta U_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta U_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$, the induced free energy $\Delta F_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta F_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$, the induced specific heat capacity $\Delta C_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta C_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$ and the induced entropy ($\Delta S_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta S_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$ for the IMMSEPM and the IMSSKPM, respectively. We have shown that the corresponding new thermodynamic quantities in 3D(NR)-NCPS symmetries (the new partition function $Z_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $Z_{nc}^{ms}(\beta,l,\Theta\{X_{nc}^{p}\})$, mean energy $U_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $U_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$, free energy $F_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $F_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$, the specific heat capacity $C_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $C_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$, and entropy $S_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $S_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$ for the IMMSEPM and the IMSSKPM, respectively, models are equal to their values in the literature (the new partition function $Z_{me}^{nl}(\beta,\nu,l)$ and $Z_{ms}^{nl}(\beta,\lambda,l)$, mean energy $U_{me}^{nl}(\beta,\nu,l)$ and $U_{ms}^{nl}(\beta,\lambda,l)$, free energy ($F_{me}^{nl}(\beta,\lambda,l)$ and $F_{\mathrm{ms}}^{nl}(\beta,\lambda,l)$), specific heat capacity ($C_{me}^{nl}(\beta,\nu,l)$ and $C_{ms}^{nl}(\beta,\lambda,l)$) and entropy ($S_{me}^{nl}(\beta,\nu,l)$ and $S_{\mathrm{ms}}^{nl}(\beta,\lambda,l)$), plus the effect of the deformation of phase-space ($\Delta Z_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta Z_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$), ($\Delta U_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta U_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$), ($\Delta F_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta F_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$), ($\Delta C_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta C_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$), and ($\Delta S_{nc}^{me}(\beta,l,\{X_{nc}^{p}\})$ and $\Delta S_{nc}^{ms}(\beta,l,\{X_{nc}^{p}\})$). It is important to note that we recover the energy equations for the SE with the modified Mobius square plus Eckart potential model and the Mobius square plus screened-Kratzer potential model in 3D(NR)-QM symmetries for the simultaneous limits ((Θ,χ,ζ) and ($\overline{\Phi},\overline{\chi},\overline{\zeta}$)) $\rightarrow (0,0,0)$, which are obtained in the main Refs. [12, 13]. After completing this successful work, we hope to handle more works, such as the improved inversely quadratic Hellmann-Kratzer potential model, the improved exponential Kratzer-Feus potential in the context of extended phase-space symmetries, and other works.

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Conflict of interest

The authors declare no conflict of interest.

Author contribution

Data availability

The authors confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

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